Of the two-level quantum mechanical system

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1 Introduction

The two-level system is the simplest quantum mechanical system and as such it is ubiquitous in quantum theory. In the context of quantum information it is called a "qubit", and the two states are accordingly denoted as $|0\rangle$ and $|1\rangle$. This is more or less realised by several systems. E.g. in quantum optics in many situations atoms can be considered as two-level systems with a ground $|g\rangle$ and an excited $|e\rangle$ state.

In the following, we mainly use the matrix representation in the basis $\{|0\rangle, |1\rangle\}$, that is

$$|0\rangle \equiv \begin{pmatrix}1\\0\end{pmatrix}, \text{ and } |1\rangle \equiv \begin{pmatrix}0\\1\end{pmatrix},$$

and accordingly

$$|0\rangle\langle 0| \equiv \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}, \text{ and } \sigma \equiv |0\rangle\langle 1| \equiv \begin{pmatrix}0 & 1\\0 & 0\end{pmatrix}, \text{ etc.}$$

2 State description—the Bloch sphere

2.1 General state

The most general two dimensional Hermitian matrix of trace one is defined by three parameters. Hence, the density matrix describing a general mixed state of the two-level system can always be written as

$$\rho \equiv \frac{1}{2}\begin{pmatrix} 1+z & x-iy \\ x+iy & 1-z \end{pmatrix} = \frac{1+x\sigma_x+y\sigma_y+z\sigma_z}{2} \equiv \frac{1+\mathbf{r}\boldsymbol{\sigma}}{2}, \tag{1}$$

where the σs are the Pauli matrices. The second equality justifies the use of $\mathbf{r} \equiv (x, y, z)$ alluding to position to parametrise the density matrix.

The remaining requirement is that ρ has to be positive semi-definite: Calculating the eigenvalues of ρ we find $\lambda_\pm = (1 \pm r)/2$, so this requirement is met if

$$r \leq 1. \tag{2}$$

Hence it is apparent that the state of a two-level system can be conveniently parametrised by the points of a sphere of unit radius, called the *Bloch sphere*. We also see that the north pole of the sphere represents the basis state $|0\rangle$, while the south pole the basis state $|1\rangle$. On the equatorial plane, we find the states with equally populated basis states. The centre of the sphere represents the maximally mixed state.

2.2 Pure states

Let us calculate the purity of the state (1):

$$\mathrm{Tr}\left\{\rho^2\right\} = \frac{1+r^2}{2}.$$

We immediately see that the pure states are characterised by $r = 1$ because this yields purity one, so that they lie *on the surface of the Bloch sphere*.

Indeed, by putting

$$|\Psi\rangle \equiv \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle, \tag{3}$$

then the parameters θ and ϕ so defined are just the well-known spherical coordinates in the Bloch-sphere representation of $|\Psi\rangle\langle\Psi|$, that is

$$\begin{aligned} z &= \cos\theta, \\ x &= \sin\theta\cos\phi, \\ y &= \sin\theta\sin\phi. \end{aligned}$$

3 Dynamics

The most general Hamiltonian acting on the system can be written as

$$H \equiv -\delta\,\sigma^\dagger\sigma - i\left(\eta\sigma^\dagger - \eta^*\sigma\right) = \begin{pmatrix} 0 & i\eta^* \\ -i\eta & -\delta \end{pmatrix}, \tag{4}$$

where we have used conventions from quantum optics. We assume a noise with reset operator σ, which in the case of an atom is just the spontaneous emission. The noise is described by the Liouvillean

$$\mathscr{L}\rho \equiv \gamma\left(2\sigma\rho\sigma^\dagger - \left[\sigma^\dagger\sigma, \rho\right]_+\right). \tag{5}$$

Together with the above Hamiltonian, they define the quantum Master equation

$$\dot{\rho} = \frac{1}{i}[H,\rho] + \mathscr{L}\rho \equiv 2\Re\left\{\frac{H_{\mathrm{nH}}}{i}\rho\right\} + 2\gamma\sigma\rho\sigma^\dagger. \tag{6}$$

We have put $\hbar = 1$. The non-Hermitian Hamiltonian

$$H_{\mathrm{nH}} \equiv H - i\gamma\,\sigma^\dagger\sigma \tag{7}$$

has been introduced.

The calculation goes as follows:

$$\frac{1}{2}\begin{pmatrix}\dot{z} & \dot{s}\\ \dot{s}^* & -\dot{z}\end{pmatrix} = \dot{\rho} = \Re\left\{\begin{pmatrix}0 & \eta^*\\ -\eta & \Omega\end{pmatrix}\begin{pmatrix}1+z & s\\ s^* & 1-z\end{pmatrix}\right\} + \gamma\begin{pmatrix}1-z & 0\\ 0 & 0\end{pmatrix} =$$

$$= \Re\left\{\begin{pmatrix}\eta^* s^* & \eta^*(1-z)\\ -\eta(1+z)+\Omega s^* & -\eta s + \Omega(1-z)\end{pmatrix}\right\} + \text{“} =$$

$$= \begin{pmatrix}\Re\{\eta^* s^*\} + \gamma(1-z) & -\eta^* z + \frac{\Omega^* s}{2}\\ -\eta z + \frac{\Omega s^*}{2} & -\Re\{\eta s\} - \gamma(1-z)\end{pmatrix}.$$

We have introduced $s \equiv x - iy$ and $\Omega \equiv -\gamma + i\delta$.

We hence obtain the following set of equations for the coordinates (the so-called Bloch equations):

$$\dot{z} = 2\Re\{\eta^* s^*\} + 2\gamma(1-z), \tag{8a}$$

$$\dot{s} = -2\eta^* z + \Omega^* s. \tag{8b}$$