Optimal bound on the quantum Fisher information

Based on few initial expectation values

Iagoba Apellaniz 1, Matthias Kleinmann 1, Otfried Gühne 2, & Géza Tóth 1,3,4

iagoba.apellaniz@gmail.com

1Department of Theoretical Physics, University of the Basque Country, Spain
2Naturwissenschaftlich-Technische Fakultät, Universität Siegen, Germany
3IKERBASQUE, Basque Foundation for Science, Spain
4Wigner Research Centre for Physics, Hungarian Academy of Sciences, Hungary

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2 QFI based on expectation values: Are they optimal?
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   - Fidelities
   - Spin-squeezed states
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4 Conclusion and outlook
● Many inequalities have been proposed to lower bound the quantum Fisher Information.

Bounds for qFI

\[
F_Q[\rho, J_z] \geq \frac{\langle J_x \rangle^2}{(\Delta J_y)^2}, \quad F_Q[\rho, J_y] \geq \beta^{-2} \frac{\langle J_x^2 + J_z^2 \rangle}{(\Delta J_z)^2 + \frac{1}{4}},
\]

\[
F_Q[\rho, J_z] \geq \frac{4(\langle J_x^2 + J_y^2 \rangle)^2}{2\sqrt{(\Delta J_x^2)^2 (\Delta J_y^2)^2} + \langle J_x^2 \rangle - 2\langle J_y^2 \rangle (1 + \langle J_x^2 \rangle) + 6\langle J_y J_x J_x J_y \rangle}
\]

[ L. Pezzé & A. Smerzi, PRL 102, 100401 (2009) ]
[ Z. Zhang & L.-M. Duan, NJP 16, 103037 (2014) ]
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The archetypical criteria that shows metrologically useful entanglement.

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The archetypical criteria that shows metrologically useful entanglement.

It is essential either to verify them or find new ones for different set of expectation values.
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The non-trivial exercise of computing the qFI

Different forms of the qFI

\[
F_Q[\rho, J_z] = 2 \sum_{\lambda, \gamma} \frac{(p_{\lambda} - p_{\gamma})^2}{p_\lambda + p_\gamma} |\langle \lambda | J_z | \gamma \rangle|^2
\]

Alternatively, as convex roof

\[
F_Q[\rho, J_z] = \min_{\{p_k, |\psi_k\rangle\}} 4 \sum_k p_k (\Delta J_z)_{|\psi_k\rangle}^2
\]

[ G. Tóth & D. Petz, PRA 87, 032324 (2013) ]
[ S. Yu, arXiv:1302.5311 ]
The non-trivial exercise of computing the qFl

- Different forms of the qFl

\[
F_Q[\rho, J_z] = 2 \sum_{\lambda, \gamma} \frac{(p_\lambda - p_\gamma)^2}{p_\lambda + p_\gamma} |\langle \lambda | J_z | \gamma \rangle|^2
\]

Alternatively, as convex roof

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F_Q[\rho, J_z] = \min \{ p_k, |\Psi_k\rangle \} 4 \sum_k p_k (\Delta J_z |_{\Psi_k})^2
\]

- For pure states it’s extremely simple

\[
F_Q[\rho, J_z] = 4 (\Delta J_z)^2
\]

[ G. Tóth & D. Petz, PRA 87, 032324 (2013) ]
[ S. Yu, arXiv:1302.5311 ]
Optimisation based on the Legendre Transform

- When \( g(\varrho) \) is a *convex roof*

\[
g(\varrho) \geq B(w := \text{Tr}[\varrho W]) = \sup_r \left( rw - \sup_{|\psi\rangle} [r \langle W \rangle - g(|\psi\rangle)] \right).
\]

Optimisation for the qFI

The *simplicity* of qFI for pure states leads to

\[ \mathcal{F}(w) = \sup_r \left( rw - \sup_{\mu} \left[ \lambda_{\text{max}}(rW - 4(J_z - \mu)^2) \right] \right). \]

For more parameters

\[ \mathcal{F}(w) = \sup_r \left( r \cdot w - \sup_{\mu} \left[ \lambda_{\text{max}}(r \cdot W - 4(J_z - \mu)^2) \right] \right). \]

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For fidelity of GHZ $\implies$ analytic solution

$$\mathcal{F} = \Theta(F_{\text{GHZ}} - 0.5)(2F_{\text{GHZ}} - 1)^2 N^2$$
Measuring $\langle J_z \rangle$ and $(\Delta J_x)^2$ for Spin Squeezed States

- 3 operators $\{J_z, J_x, J_x^2\}$
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- 3 operators $\{J_z, J_x, J_x^2\}$
- Reducing one dimension of $\mathcal{F}$ on the $\langle J_x \rangle$ direction

$$\mathcal{F} \geq \mathcal{F}(\langle J_x \rangle = 0)$$

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$$\mathcal{F}(\langle J_z \rangle, (\Delta J_x)^2) := \mathcal{F}(\langle J_z \rangle, \langle J_x^2 \rangle)$$
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- Pezze-Smerzi bound, $F_Q \geq \langle J_z \rangle^2 / (\Delta J_x)^2$, can be verified.
**4-particle system**

**Left:** For $(\Delta J_x)^2 < 1.5$ it almost coincides with the P-S bound $F_Q \geq \langle J_z \rangle^2 / (\Delta J_x)^2$. **Right:** The measurement of $\langle J_x^4 \rangle$ improves the bound.

Scaling the result for large systems


\[ N = 2300 \quad \xi_s^2 = -8.2\text{dB} = 0.1514 \]
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- We choose

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\[ N = 2300 \quad \xi_s^2 = -8.2\,\text{dB} = 0.1514 \]

- We choose \[ \langle J_z \rangle = 0.85 \frac{N}{2} \]
- P-S bound results is \[ \frac{F_Q}{N} \geq \frac{1}{\xi_s^2} = 6.605 \]
Starting from small systems, and assuming bosonic symmetry.

The results obtained with our method converge to P-S bound!
Metrology with unpolarised Dicke states

- 3 operators \( \{ J^2_x, J^2_y, J^2_z \} \); Experimental constraint:
  \[ \langle J^2_x \rangle = \langle J^2_y \rangle. \]
3 operators $\{J_x^2, J_y^2, J_z^2\}$; Experimental constraint:

$$\langle J_x^2 \rangle = \langle J_y^2 \rangle.$$

For $\sum_i \langle J_i^2 \rangle = \frac{N}{2} (\frac{N}{2} + 1)$, i.e. bosonic symmetry, and 6-particle system:
Realistic characterisation of Dicke state

Experiment → [B. Lücke et al., PRL 112, 155304 (2014)]

\[ N = 7900 \quad \langle J_z^2 \rangle = 112 \pm 31 \]

\[ \langle J_x^2 \rangle = \langle J_y^2 \rangle = 6 \times 10^6 \pm 0.6 \times 10^6 \]

- For that large system, we start from small ones similar to the spin-squeezed states.
Numerical lower bound.

Similarly to the spin-squeezed states, the bound *converges quickly.*

We prove that for realistic cases *the optimisation is feasible.*
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Conclusion and Outlook

1. We prove that for realistic cases *the optimisation is feasible*.
2. We used *our approach to verify* that the P-S bound is tight.
3. We have shown that the lower bounds can be *improved with extra constraints*. 
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4. For large systems the optimisation method can be complemented with scaling considerations.
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1. We prove that for realistic cases *the optimisation is feasible*.

2. We used *our approach to verify* that the P-S bound is tight.

3. We have shown that the lower bounds can be *improved with extra constraints*.

4. For large systems the *optimisation method can be complemented* with scaling considerations.

5. The *method very versatile* and it can be used in many other situations.
Thank you for your attention!

Preprint → arXiv:1511.05203

Groups’ home pages
→ https://sites.google.com/site/gedentqopt
→ http://www.physik.uni-siegen.de/tqo/

iagoba  matthias  otfried  géza