How long does it take to obtain a physical density matrix?

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Outline

1. **Motivation**
   - Why quantum tomography is important?

2. **Quantum experiments with multi-qubit systems**
   - Physical systems
   - Local measurements

3. **Full quantum state tomography**
   - Basic ideas and scaling
   - Experiments
   - Approaches to solve the scalability problem

4. **How to obtain a density matrix**

5. **Extra slides**
Many experiments aiming to create many-body entangled states.

Quantum state tomography can be used to check how well the state has been prepared.

However, the number of measurements scales exponentially with the number of qubits.
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5 Extra slides
Physical systems

State-of-the-art in experiments

- 14 qubits with trapped cold ions

- 10 qubits with photons
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A single local measurement setting is the basic unit of experimental effort. A local setting means measuring operator $A^{(k)}$ at qubit $k$ for all qubits.

All two-qubit, three-qubit correlations, etc. can be obtained.

$$\langle A^{(1)} A^{(2)} \rangle, \langle A^{(1)} A^{(3)} \rangle, \langle A^{(1)} A^{(2)} A^{(3)} \rangle, \ldots$$
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Full quantum state tomography

- The density matrix can be reconstructed from $3^N$ measurement settings.

**Example**

For $N = 4$, the measurements are

1. X X X X X X
2. X X X X Y
3. X X X Z
   ... 
3^4. Z Z Z Z Z Z

- Note again that the number of measurements scales exponentially in $N$. 
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Approaches to solve the scalability problem

- If the state is expected to be of a certain form (MPS), we can measure the parameters of the ansatz.

- If the state is of low rank, we need fewer measurements.

- We make tomography in a subspace of the density matrices (our approach).
Obtain a density matrix

- The density matrix can be decomposed into correlations as
  \[ \rho = \frac{1}{2^n} \sum_{\mu} T_{\mu} \sigma_{\mu}, \]
  where \( \sigma_{\mu} = \sigma_{\mu_1} \otimes \sigma_{\mu_2} \otimes \cdots \otimes \sigma_{\mu_n}, \mu_i \in \{0, 1, 2, 3\}, \) and \( \sigma_0 \) denotes the identity matrix.

- The correlation matrix is defined as \( T_{\mu} = \langle \sigma_{\mu} \rangle. \)

- How can we obtain the estimate \( \tilde{\rho} \)? We just measure \( T_{\mu}. \)
How can we obtain the estimate $\tilde{\varrho}$? We just measure $T_\mu = \langle \sigma_\mu \rangle$.

Problem: we have finite number of measurements.
 Obtain a density matrix III

- 1 qubit, 11 measurements.

\[ \langle \sigma_Z \rangle = +1 \]
\[ \langle \sigma_Z \rangle = -1 \]
\[ \langle \sigma_X \rangle = -1 \] \[ \langle \sigma_X \rangle = +1 \]

Why negative eigenvalues are a problem?

- We cannot calculate fidelities with a mixed state, entropies, purity, entanglement, etc.
- We can still calculate the fidelity with a pure state. This is just the expectation value of a projector.
Fitting

- Method to get rid of the negative eigenvalues of $\rho$.
- Find the physical density matrix in a best agreement with the experimental data.
- Main methods: maximum likelihood, least squares.
Problems with fitting

- Fidelity changes, bias, detection of fake entanglement

[Schwemmer et al., PRL 114, 080403 (2015).]
Problems with fitting

Before

After

Small eigenvalues increase  Large eigenvalues decrease
Let us analyze the problem

- Completely mixed state

\[ \rho_{wn} = \frac{1}{2^n} \sigma_{0,0,\ldots,0} = \frac{1}{2^n} I \]

with \(2^n\) degenerate eigenvalues \(\lambda_i = 1/2^n\).

- We use overcomplete tomography, which is based on measuring the Pauli correlations.
Distribution of eigenvalues

- Consider $n = 6$ qubit maximally mixed state
- Simulate $N = 100$ measurements per setting
- Estimate density matrix
- Repeat 10 000 times
- Histogram of eigenvalues
Consider \( n = 6 \) qubit maximally mixed state
Simulate \( N = 100 \) measurements per setting
Estimate density matrix
Repeat 10 000 times
Histogram of eigenvalues
How long do we have to measure to get a physical state?

- Pure state mixed with white noise
  \[ \varrho_q = q|\psi\rangle\langle\psi| + (1 - q)\varrho_{\text{cm}}. \]

- The center is shifted to
  \[ c_q = \frac{1 - q}{2^n - r}. \]

- The radius is
  \[ R = 2\sqrt{\frac{10^n - 1}{12^n}} \frac{1}{\sqrt{N}} \approx 2 \left( \frac{5}{6} \right)^{\frac{n}{2}} \frac{1}{\sqrt{N}}. \]

- Physical \( \varrho \) if
  \[ R \leq c_q \Rightarrow N \geq N_0 = 4 \left( \frac{5}{6} \right)^n \left( \frac{2^n - 1}{1 - q} \right)^2. \]
How long do we have to measure to get a physical state? II

The minimum number of measurements needed is

\[ N_0 = 4 \left( \frac{5}{6} \right)^n \left( \frac{2^n - 1}{1 - q} \right)^2. \]
How long do we have to measure to get a physical state? III

- Six-qubit GHZ state mixed with $q = 0.2$ white noise
Other type of tomography

- Not all tomographies lead to a Wigner semicircle
Hypothesis testing

- We prepare a six-qubit Dicke state

\[ |D_6^{(3)}\rangle = \frac{1}{\sqrt{6}}(|000111\rangle + |001011\rangle + \ldots + |111000\rangle). \]

- Quantum state tomography with around 230 events per setting.

- Hypothesis: 3 eigenvalues + noise. Is this correct?
We prepare a six-qubit Dicke state

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Hypothesis testing III

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Quantum state tomography with around 230 events per setting.

Hypothesis: 3 eigenvalues + noise. Is this correct?
Is the hypothesis correct?

- Empirical distribution function (EDF) vs. Cumulative distribution function (CDF) of the Wigner semicircle
Our method

Before

After

Small eigenvalues are replaced by their average
Large eigenvalues do not change
Just to compare: old method

Before

$\lambda_k$

0

.....

After

$\lambda_k$

0

1

Small eigenvalues increase  Large eigenvalues decrease
Summary

- We discussed the distribution of the eigenvalues of density matrices obtained from tomography.
- We suggested a method to get rid of negative eigenvalues.
- I thank Lukas Knips for most of the figures for this talk.

See:
L. Knips, C. Schwemmer, N. Klein, J. Reuter, G. Tóth, and H. Weinfurter,
How long does it take to obtain a physical density matrix?, arxiv:1512.06866.

THANK YOU FOR YOUR ATTENTION!
• Concept: compare moments of eigenvalue distribution to moments of ideal semicircle function

• Define semicircle distribution

\[ f_{c,R}(x) = \frac{2}{\pi R^2} \sqrt{(x - c)^2 - R^2} \]

with (even) moments

\[
m^2_{2k} = \int_{-\infty}^{\infty} f_{0,R}(x) x^{2k} \, dx = \frac{R^{2k}}{2}.
\]

\[
m^4_{4k} = \int_{-\infty}^{\infty} f_{0,R}(x) x^{4k} \, dx = 2 \left( \frac{R}{2} \right)^4.
\]

\[
m^6_{4k} = \int_{-\infty}^{\infty} f_{0,R}(x) x^{6k} \, dx = 5 \left( \frac{R}{2} \right)^6.
\]

\[
m^8_{8k} = \int_{-\infty}^{\infty} f_{0,R}(x) x^{8k} \, dx = 14 \left( \frac{R}{2} \right)^8.
\]

Using the Catalan numbers

\[ C_{j+1} = C_j \frac{2(2j + 1)}{j + 2} \]

we obtain

\[ m^2_{2k} = \int_{-\infty}^{\infty} f_{0,R}(x) x^{2k} \, dx = C_k \left( \frac{R}{2} \right)^{2k} \]

• Odd (centralized) moments vanish

• Goal: reproduce Catalan numbers in distribution of eigenvalues

• Calculate all moments of eigenvalue distribution:

\[ m^2_{k} = \frac{1}{2n} \sum_{i=1}^{2n} \mathbb{E} [\lambda_i^k] \]

\[ = \frac{1}{2n} \mathbb{E} \left[ \sum_{i=1}^{2n} \lambda_i^k \right] \]

\[ = \mathbb{E} \left[ \frac{1}{2n} \text{Tr} \left( \mathbf{D}^k \right) \right] \]

\[ = \mathbb{E} \left[ \frac{1}{2n} \text{Tr} \left( \left( \mathbf{U}^\dagger \mathbf{gU} \right)^k \right) \right] \]

\[ = \mathbb{E} \left[ \frac{1}{2n} \text{Tr} \left( \mathbf{d}^k \right) \right] \]

• Fourth moment:

\[ m^4_{k} = \frac{1}{2n} \sum_{i=1}^{2n} \mathbb{E} [\lambda_i^4] \]

\[ = \frac{1}{2n} \sum_{\mu, \nu, \tilde{\mu}, \tilde{\nu}} \mathbb{E} \left[ T_{\mu\tilde{\mu}} T_{\nu\tilde{\nu}} \right] \cdot \text{Tr} \left( \mathbf{g} \right) \]

\[ = \frac{1}{2n} \sum_{\mu, \nu, \tilde{\mu}, \tilde{\nu}} \mathbb{E} \left[ T_{\mu\tilde{\mu}} T_{\nu\tilde{\nu}} \right] \cdot \text{Tr} \left( \sum_{i=1}^{6} P_i \left( \sigma_i \sigma_i \sigma_i \sigma_i \right) \right) \]

• Sixth moment:

\[ m^6_{6} = \frac{1}{2n} \sum_{i=1}^{2n} \mathbb{E} [\lambda_i^6] \]

\[ \approx \frac{1}{2n} \sum_{\mu, \nu, \tilde{\mu}, \tilde{\nu}} \mathbb{E} \left[ T_{\mu\tilde{\mu}} T_{\nu\tilde{\nu}} \right] \cdot \text{Tr} \left( \sum_{i=1}^{90} P_i \left( \sigma_i \sigma_i \sigma_i \sigma_i \sigma_i \sigma_i \right) \right) \]

• Second moment of (centered) distribution:

\[ m^2_{2} = \frac{1}{2n} \sum_{\mu, \nu, \tilde{\mu}, \tilde{\nu}} \mathbb{E} \left[ T_{\mu\tilde{\mu}} T_{\nu\tilde{\nu}} \right] 2^n \delta_{\mu, \nu} \]

\[ = \frac{2^n}{2n} \sum_{\tilde{\mu}} \mathbb{E} \left[ T_{\tilde{\mu}\tilde{\mu}} \right] \]

overcomplete Pauli scheme:

\[ m^2_{2} = \frac{1}{4^n N} \sum_{j=0}^{n-1} \left( \frac{n}{j} \right) \frac{3^n - j}{3^j} \]

\[ = \frac{10^n - 1}{12n} \]

with n qubits, N events per basis element.

• Comparison of \( m^2_{2k, \text{even}} \) yields:

\[ R = 2 \sqrt{\frac{10^n - 1}{12n}} \]

• Only non-crossing partitions (amount given by Catalan numbers) contribute:

\[ m^2_{k} = \frac{1}{2n} \sum_{i=1}^{2n} \mathbb{E} [\lambda_i^{2k}] = C_k \frac{1}{N^k} \]