Tutorial: Quantum metrology from a quantum information science perspective


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1 Motivation
   • Why is quantum metrology interesting?

2 Simple examples of quantum metrology
   • Magnetometry with the fully polarized state
   • Magnetometry with the spin-squeezed state
   • Metrology with the GHZ state
   • Dicke states
   • Singlet states

3 Entanglement theory
   • Multipartite entanglement
   • The spin-squeezing criterion

4 Quantum metrology using the quantum Fisher information
   • Quantum Fisher information
   • Quantum Fisher information in linear interferometers
Why is quantum metrology interesting?

- Recent technological development has made it possible to realize large coherent quantum system, i.e., in cold gases.

- Can such quantum systems outperform classical systems?

- The problem can be understood better based on entanglement theory.
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Classical case: arbitrary precision ("in principle").
Magnetometry with the fully polarized state

- Let us see the quantum case.

- $N$ spin-1/2 particles, all fully polarized in the $z$ direction.

- Magnetic field $B$ points to the $y$ direction.

- Note the uncertainty ellipses. $\Delta \theta_{fp}$ is the minimal angle difference we can measure.
Collective angular momentum components

\[ J_l := \sum_{n=1}^{N} j_l^{(n)} \]

for \( l = x, y, z \), where \( j_l^{(n)} \) are single particle operators.

Dynamics

\[ U_\theta = e^{-iJ_y \theta}, \]

where \( \hbar = 1 \), and the angle \( \theta \) is

\[ \theta = \gamma B t, \]

where \( \gamma \) is the gyromagnetic ratio, and \( t \) is the time.
Measure an operator $M$ to get the estimate $\theta$.

The precision is given by the error propagation formula

$$(\Delta \theta)^2 = \frac{(\Delta M)^2}{|\partial_\theta \langle M \rangle|^2}.$$
Magnetometry with the fully polarized state V

- We measure the operator
  \[ M = J_x. \]

- Expectation value and variance
  \[
  \langle M \rangle(\theta) = \langle J_z \rangle \sin(\theta) + \langle J_x \rangle \cos(\theta),
  \]
  \[
  (\Delta M)^2(\theta) = (\Delta J_x)^2 \cos^2(\theta) + (\Delta J_z)^2 \sin^2(\theta)
  + \left( \frac{1}{2} \langle J_x J_z + J_z J_x \rangle - \langle J_x \rangle \langle J_z \rangle \right) \sin(2\theta).
  \]

- Using \( \langle J_x \rangle = 0 \), in the \( \theta \to 0 \) limit
  \[
  (\Delta \theta)^2 = \frac{(\Delta M)^2}{|\partial_\theta \langle M \rangle|^2} = \frac{(\Delta J_x)^2}{\langle J_z \rangle^2} = \frac{1}{N}.
  \]
It is not like a classical clock arm, we have a nonzero uncertainty

$$(\Delta \theta)^2 = \frac{1}{N}.$$
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Magnetometry with the spin-squeezed state

- We can increase the precision by spin squeezing

\[ \Delta \theta_{\text{fp}} \text{ and } \Delta \theta_{\text{sq}} \text{ are the minimal angle difference we can measure.} \]

We can reach

\[ (\Delta \theta)^2 < \frac{1}{N}. \]
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Greenberger-Horne-Zeilinger (GHZ) state

\[ |\text{GHZ}_N\rangle = \frac{1}{\sqrt{2}} (|0\rangle^\otimes N + |1\rangle^\otimes N) , \]

Unitary

\[ U_\theta = e^{-iJ_z \theta} . \]

Dynamics

\[ |\text{GHZ}_N\rangle(\theta) = \frac{1}{\sqrt{2}} (|0\rangle^\otimes N + e^{-iN\theta} |1\rangle^\otimes N) , \]
Metrology with the GHZ state II

- We measure

\[ M = \sigma_x^\otimes N, \]

which is the parity in the \( x \)-basis.

- Expectation value and variance

\[ \langle M \rangle = \cos(N\theta), \quad (\Delta M)^2 = \sin^2(N\theta). \]

- For \( \theta \approx 0 \), the precision is

\[ (\Delta \theta)^2 = \frac{(\Delta M)^2}{|\partial_\theta \langle M \rangle|^2} = \frac{1}{N^2}. \]

We reached the Heisenberg-limit

\[(\Delta \theta)^2 = \frac{1}{N^2}.\]

The fully polarized state reached only the shot-noise limit

\[(\Delta \theta)^2 = \frac{1}{N}.\]
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Symmetric Dicke states with $\langle J_z \rangle = 0$ (simply “Dicke states” in the following) are defined as

$$|D_N\rangle = \left( \frac{N}{N/2} \right)^{-\frac{1}{2}} \sum_k \mathcal{P}_k \left( |0\rangle^{N/2} \otimes |1\rangle^{N/2} \right).$$

E.g., for four qubits they look like

$$|D_4\rangle = \frac{1}{\sqrt{6}} \left( |0011\rangle + |0101\rangle + |1001\rangle + |0110\rangle + |1010\rangle + |1100\rangle \right).$$
For our symmetric Dicke state

\[ \langle J_l \rangle = 0, \ l = x, y, z, \ \langle J_2^z \rangle = 0, \ \langle J_2^x \rangle = \langle J_2^y \rangle = \text{large}. \]

Linear metrology

\[ U = \exp(-iJ_y \theta). \]

Measure \( \langle J_2^z \rangle \) to estimate \( \theta \). (We cannot measure first moments, since they are zero.)
Dicke states are more robust to noise than GHZ states.

Dicke states can also reach the Heisenberg-scaling like GHZ states.


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   - Singlet states

3. **Entanglement theory**
   - Multipartite entanglement
   - The spin-squeezing criterion

4. **Quantum metrology using the quantum Fisher information**
   - Quantum Fisher information
   - Quantum Fisher information in linear interferometers
Metrology with the singlet state

- For our singlet state
  
  \[ \langle J_l \rangle = 0, \quad \langle J_l^2 \rangle = 0, \quad l = x, y, z, \]

- Invariant under the actions of homogeneous magnetic fields, i.e., operations of the type \( \exp(-iJ_{\vec{n}}\theta) \).

- Sensitive to gradients.

- We do not need to measure the homogeneous field, if we want to estimate the gradient.

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A state is (fully) separable if it can be written as

$$\sum_k p_k \varrho_k^{(1)} \otimes \varrho_k^{(2)} \otimes \ldots \otimes \varrho_k^{(N)}.$$ 

If a state is not separable then it is entangled (Werner, 1989).
A pure state is **$k$-producible** if it can be written as

$$|\Phi\rangle = |\Phi_1\rangle \otimes |\Phi_2\rangle \otimes |\Phi_3\rangle \otimes |\Phi_4\rangle \ldots$$

where $|\Phi_l\rangle$ are states of at most $k$ qubits.

A mixed state is $k$-producible, if it is a mixture of $k$-producible pure states.

[ e.g., Gühne, GT, NJP 2005. ]

If a state is not $k$-producible, then it is at least $(k + 1)$-particle entangled.
$k$-producibility/$k$-entanglement II

Separable

2-producible

$(N-1)$-producible

$N$-producible

...
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The standard spin-squeezing criterion

Spin squeezing criteria for entanglement detection

\[ \xi_s^2 = N \frac{(\Delta J_x)^2}{\langle J_y \rangle^2 + \langle J_z \rangle^2}. \]

If \( \xi_s^2 < 1 \) then the state is entangled.

[Sørensen, Duan, Cirac, Zoller, Nature (2001).]

States detected are like this:

- Variance of \( J_x \) is small
- \( J_z \) is large
Larger and larger multipartite entanglement is needed to larger and larger squeezing ("extreme spin squeezing").

\[ N = 100 \text{ spin-1/2 particles}, \ J_{\text{max}} = \frac{N}{2}. \]

The full set of entanglement criteria with collective observables has been obtained.

One of these criteria is the following

\[ \langle J_x^2 \rangle + \langle J_y^2 \rangle \leq (N - 1)(\Delta J_z)^2 + \frac{N}{2}. \]

It detects entanglement close to Dicke states.

Multipartite entanglement detection around Dicke states

- Generalized spin squeezing inequality. BEC, 8000 particles. 28-particle entanglement is detected.

\[ J_{\text{eff}}^2 = J_x^2 + J_y^2 \] and \( J_{\text{max}} = N/2 \).

As we have said, the full set of entanglement criteria with collective observables has been obtained.

Another one of these criteria is the following

\[(\Delta J_x)^2 + (\Delta J_y)^2 + (\Delta J_z)^2 \geq \frac{N}{2}\].

It detects entanglement close to singlet states.

Singlets

For separable states of $N$ spin-$j$ particles

$$\xi^2_{\text{singlet}} = \frac{(\Delta J_x)^2 + (\Delta J_y)^2 + (\Delta J_z)^2}{Nj} \geq 1.$$ 

For the singlet

$$(\Delta J_x)^2 + (\Delta J_y)^2 + (\Delta J_z)^2 = 0, \quad \xi^2_{\text{singlet}} = 0.$$ 

Number of particles entangled with the rest

$$N_e \geq N(1 - \xi^2_{\text{singlet}}).$$


Our experience so far

- We looked at various setups.
- We find that better precision needs more entanglement.
- Question: Is this general?
- Answer: Yes.
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Quantum metrology

• Fundamental task in metrology

\[ U(\theta) = \exp(-iA\theta) \]

We have to estimate $\theta$ in the dynamics

\[ U = \exp(-iA\theta). \]
Precision of parameter estimation (slide repeated)

- Measure an operator $M$ to get the estimate $\theta$.

- The precision is given by the error propagation formula

\[
(\Delta\theta)^2 = \frac{(\Delta M)^2}{|\partial_{\theta} \langle M \rangle|^2}.
\]
The quantum Fisher information

Cramér-Rao bound on the precision of parameter estimation

For every $M$

$$(\Delta \theta)^2_M \geq \frac{1}{F_Q[\varrho, A]} ,$$

where $F_Q[\varrho, A]$ is the quantum Fisher information.

- The bound is even more general, includes any estimation strategy, even POVM's.

- The quantum Fisher information is

$$F_Q[\varrho, A] = 2 \sum_{k,l} \frac{(\lambda_k - \lambda_l)^2}{\lambda_k + \lambda_l} |\langle k|A|l \rangle|^2 ,$$

where $\varrho = \sum_k \lambda_k |k\rangle \langle k|$. 
The optimal measurement

An optimal measurement can be carried out if we measure in the eigenbasis of the symmetric logarithmic derivative $L$ given as

$$L = 2i \sum_{k,l} \frac{\lambda_k - \lambda_l}{\lambda_k + \lambda_l} |k\rangle\langle l| |k\rangle\langle l| |A| |l\rangle,$$

where $\rho = \sum_k \lambda_k |k\rangle\langle k|.$

- $L$ is defined by
  $$\frac{d\rho_\theta}{d\theta} = \frac{1}{2} (L\rho_\theta + \rho_\theta L).$$

- Unitary dynamics with the Hamiltonian $A$
  $$\frac{d\rho_\theta}{d\theta} = i (\rho_\theta A - A\rho_\theta).$$

- Hence, the formula above can be obtained.

- Relation to the QFI: $F_Q[\rho, A] = \text{Tr}(L^2 \rho).$
Multi-parameter estimation

The Cramér-Rao bound for the multi-parameter case is

$$C - F^{-1} \geq 0.$$  

- $C$ is now the covariance matrix with elements
  $$C_{mn} = \langle \theta_m \theta_n \rangle - \langle \theta_m \rangle \langle \theta_n \rangle.$$

- $F$ is the Fisher matrix
  $$F_{mn} \equiv F_Q[\varrho, A_m, A_n] = 2 \sum_{k,l} \frac{(\lambda_K - \lambda_l)^2}{\lambda_K + \lambda_l} \langle k | A_m | l \rangle \langle l | A_n | k \rangle,$$
  where $\varrho = \sum_k \lambda_k | k \rangle \langle k |$. 


The quantum Fisher information appears in the Taylor expansion of $F_B$:

$$F_B(\varrho, \varrho_\theta) = 1 - \theta^2 \frac{F_Q[\varrho, A]}{4} + \mathcal{O}(\theta^3),$$

where

$$\varrho_\theta = \exp(-iA\theta)\varrho \exp(+iA\theta).$$

- Bures fidelity

$$F_B(\varrho_1, \varrho_2) = \text{Tr} \left( \sqrt{\sqrt{\varrho_1 \varrho_2} \sqrt{\varrho_1}} \right)^2.$$

- Clearly,

$$0 \leq F_B(\varrho_1, \varrho_2) \leq 1.$$

The fidelity is 1 only if $\varrho_1 = \varrho_2$. 
For pure states, it equals four times the variance,

$$F_Q[|\psi\rangle, A] = 4(\Delta A)^2_\psi.$$ 

For mixed states, it is convex

$$F_Q[\rho, A] \leq \sum_k p_k F_Q[|\psi_k\rangle, A],$$

where

$$\rho = \sum_k p_k |\psi_k\rangle \langle \psi_k|.$$
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The Hamiltonian $A$ is defined as

$$A = J_l = \sum_{n=1}^{N} j_l^{(n)}, \quad l \in \{x, y, z\}.$$ 

There are no interaction terms.

The dynamics rotates all spins in the same way.
The quantum Fisher information vs. entanglement

- For separable states
  \[ F_Q[\rho, J_l] \leq N, \quad l = x, y, z. \]

- For states with at most \( k \)-particle entanglement (\( k \) is divisor of \( N \))
  \[ F_Q[\rho, J_l] \leq kN. \]

- Macroscopic superpositions (e.g, GHZ states, Dicke states)
  \[ F_Q[\rho, J_l] \propto N^2, \]
The quantum Fisher information vs. entanglement

5 spin-1/2 particles

\( F_Q \)

At least

- 5-entanglement
- 4-entanglement
- 3-entanglement
- 2-entanglement
Let us use the Cramér-Rao bound

- For separable states
  \[(\Delta \theta)^2 \geq \frac{1}{N}, \quad l = x, y, z.\]


- For states with at most \(k\)-particle entanglement (\(k\) is divisor of \(N\))
  \[(\Delta \theta)^2 \geq \frac{1}{kN}.\]


- Macroscopic superpositions (e.g., GHZ states, Dicke states)
  \[(\Delta \theta)^2 \propto \frac{1}{N^2},\]

Noisy metrology: Simple example

- A particle with a state $\varrho_1$ passes through a map that turns its internal state to the fully mixed state with some probability $p$ as

  \[ \varepsilon_p(\varrho_1) = (1 - p)\varrho_1 + p\frac{1}{2}. \]

- This map acts in parallel on all the $N$ particles

  \[ \varepsilon_p^\otimes N(\varrho) = \sum_{n=0}^{N} p_n\varrho_n, \]

  where the state obtained after $n$ particles decohered into the completely mixed state is

  \[ \varrho_n = \frac{1}{N!} \sum_k \Pi_k \left[ \left(\frac{1}{2}\right)^\otimes n \otimes \text{Tr}_{1,2,...,n}(\varrho) \right] \Pi_k^+. \]

  The summation is over all permutations $\Pi_k$. The probabilities are

  \[ p_n = \binom{N}{n} p^n (1 - p)^{(N-n)}. \]
Rewriting it

\[ \epsilon_p^N(\varrho) = \sum_{n=0}^{N} p_n \frac{1}{N!} \sum_k \Pi_k \left[ \left( \frac{1}{2} \right)^{\otimes n} \otimes \text{Tr}_{1,2,\ldots,n}(\varrho) \right] \Pi_k^\dagger. \]

For the noisy state

\[ (\Delta J_x)^2 \geq \sum_n p_n (\Delta J_x)^2_{\varrho_n} \geq \sum_n p_n \frac{n}{4} = \frac{pN}{4}. \]

Hence, for the precision shot-noise scaling follows

\[ (\Delta \theta)^2 = \frac{(\Delta J_x)^2}{\langle J_z \rangle^2} \geq \frac{pN}{4N^2/4} \propto \frac{1}{N}. \]
In the most general case, uncorrelated single particle noise leads to shot-noise scaling after some particle number.

Figure from [R. Demkowicz-Dobrzański, J. Kołodyński, M. Guță, Nature Comm. 2012.]

Correlated noise is different.
Reviews


Summary

- We reviewed quantum metrology from a quantum information point of view.

See:

Géza Tóth and Iagoba Apellaniz,
Quantum metrology from a quantum information science perspective,

THANK YOU FOR YOUR ATTENTION!