Entanglement between two spatially separated atomic modes

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1 Motivation
- Why entanglement is important?

2 Spin squeezing and entanglement
- Entanglement
- Collective measurements
- The original spin-squeezing criterion
- Generalized criteria for $j = \frac{1}{2}$

3 Dicke states
- Detection of multipartite entanglement close to Dicke states
- Dicke state in double well
Why multipartite entanglement is important?

- Many experiments are aiming to create entangled states with many atoms.

- Full tomography is not possible, we still have to say something meaningful.

- Only collective quantities can be measured.

- Thus, entanglement detection seems to be a good idea ...
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A state is (fully) separable if it can be written as

$$\sum_k p_k \rho^{(k)}_1 \otimes \rho^{(k)}_2 \otimes \ldots \otimes \rho^{(k)}_N.$$ 

If a state is not separable then it is entangled.
A pure state is $k$-producible if it can be written as

$$|\Phi\rangle = |\Phi_1\rangle \otimes |\Phi_2\rangle \otimes |\Phi_3\rangle \otimes |\Phi_4\rangle \ldots$$

where $|\Phi_i\rangle$ are states of at most $k$ qubits.

A mixed state is $k$-producible, if it is a mixture of $k$-producible pure states.

[ e.g., O. Gühne and GT, New J. Phys 2005. ]

- If a state is not $k$-producible, then it is at least $(k + 1)$-particle entangled.
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For spin-$\frac{1}{2}$ particles, we can measure the collective angular momentum operators:

$$J_l := \frac{1}{2} \sum_{k=1}^{N} \sigma_l^{(k)},$$

where $l = x, y, z$ and $\sigma_l^{(k)}$ a Pauli spin matrices.

We measure the expectation values $\langle J_l \rangle$.

We can also measure the variances

$$(\Delta J_l)^2 := \langle J_l^2 \rangle - \langle J_l \rangle^2.$$
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The standard spin-squeezing criterion

The spin squeezing criterion for entanglement detection is

$$\xi_s^2 = N \frac{(\Delta J_z)^2}{\langle J_x \rangle^2 + \langle J_y \rangle^2}.$$ 

[A. Sørensen, L.M. Duan, J.I. Cirac, P. Zoller, Nature 409, 63 (2001).]

- If $\xi_s^2 < 1$ then the state is entangled.
- States detected are like this:

  - $J_x$ is large
  - Variance of $J_z$ is small

- They are good for metrology!
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Let us assume that for a system we know only
\[
\vec{J} := (\langle J_x \rangle, \langle J_y \rangle, \langle J_z \rangle),
\]
\[
\vec{K} := (\langle J_x^2 \rangle, \langle J_y^2 \rangle, \langle J_z^2 \rangle).
\]

Then any state violating the following inequalities is entangled:
\[
\langle J_x^2 \rangle + \langle J_y^2 \rangle + \langle J_z^2 \rangle \leq \frac{N(N+2)}{4},
\]
\[
(\Delta J_x)^2 + (\Delta J_y)^2 + (\Delta J_z)^2 \geq \frac{N}{2},
\]
\[
\langle J_k^2 \rangle + \langle J_l^2 \rangle \leq (N - 1)(\Delta J_m)^2 + \frac{N}{2},
\]
\[
(N - 1) \left[ (\Delta J_k)^2 + (\Delta J_l)^2 \right] \geq \langle J_m^2 \rangle + \frac{N(N-2)}{4},
\]
where \(k, l, m\) take all the possible permutations of \(x, y, z\).

Generalized spin squeezing criteria for $j = \frac{1}{2}$

- Separable states are in the polytope

- We set $\langle J_l \rangle = 0$ for $l = x, y, z$. 
Spin squeezing criteria – Two-particle correlations

All quantities depend only on two-particle correlations

\[ \langle J_l \rangle = N \langle j_l \otimes \mathbb{I} \rangle_{\rho_{p2}}; \quad \langle J_l^2 \rangle = \frac{N}{4} + N(N - 1) \langle j_l \otimes j_l \rangle_{\rho_{p2}}. \]

- Average 2-particle density matrix

\[ \rho_{2p} = \frac{1}{N(N - 1)} \sum_{n \neq m} \rho_{mn}. \]

- We can detect states with a separable \( \rho_{p2} \).

- We can even detect multipartite entanglement!
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Netflix movie “Spectral”
Filmed in Budapest

Bose-Einstein condensate
people
Dicke states

- Dicke states: eigenstates of $\vec{J}^2 = J_x^2 + J_y^2 + J_z^2$ and $J_z$.

- Symmetric Dicke states with $\langle J_z \rangle = \langle J_z^2 \rangle = 0$

$$|D_N\rangle = \left(\frac{N}{N/2}\right)^{-1/2} \sum_k \mathcal{P}_k \left( |0\rangle^{\otimes N/2} \otimes |1\rangle^{\otimes N/2} \right).$$

Due to symmetry, $\langle \vec{J}^2 \rangle$ is maximal.

- E.g., for four qubits they look like

$$|D_4\rangle = \frac{1}{\sqrt{6}} \left( |0011\rangle + |0101\rangle + |1001\rangle + |0110\rangle + |1010\rangle + |1100\rangle \right).$$

[photons: Kiesel et al., PRL 2007; Prevedel et al., PRL 2007; Wieczorek et al., PRL 2009]
[cold atoms: Lücke et al., Science 2011; Hamley et al., Science 2011]
Dicke states are useful because they ...

- ... possess strong multipartite entanglement, like GHZ states.
  [GT, JOSAB 2007.]

- ... are optimal for quantum metrology, similarly to GHZ states.
  [Hyllus et al., PRA 2012; Lücke et al., Science 2011.]

- ... are macroscopically entangled, like GHZ states.
  [Fröwis, Dür, PRL 2011]
Let us rewrite the third inequality

\[ \langle J_x^2 \rangle + \langle J_y^2 \rangle - \frac{N}{2} \leq (N - 1)(\Delta J_z)^2. \]

It detects states close to Dicke states since

\[ \langle J_x^2 + J_y^2 \rangle = \frac{N}{2} \left( \frac{N}{2} + 1 \right) = \text{max.}, \]

\[ \langle J_z^2 \rangle = 0. \]

"Pancake" like uncertainty ellipse.
Multipartite entanglement

- Bose-Einstein condensate, 8000 particles. 28-particle entanglement is detected.

\[ J_{\text{eff}}^2 = J_x^2 + J_y^2, \quad J_{\text{max}} = \frac{N}{2}. \]

[Lücke et al., PRL 112, 155304 (2014).]
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Bipartite entanglement from bosonic multipartite entanglement

- In the BEC, "all the particles are at the same place."
- In the usual formulation, entanglement is between spatially separated parties.
- Is multipartite entanglement within a BEC useful/real?
- Answer: yes!
Dilute cloud argument

\[ |n_0 = 1\rangle |n_1 = 1\rangle \]

\[ \frac{1}{\sqrt{2}} (|01\rangle + |10\rangle) \]
Splitting of the ensembles: after splitting into two, we have bipartite entanglement if we had before multipartite entanglement.

The splitting does not generate entanglement, if we consider projecting to a fixed particle number.

Experiment in the group of Carsten Klempt at the University of Hannover

- Rubidium BEC, spin-1 atoms.
- Initially all atoms in state $|0\rangle$.
- Dynamics
  $$H = a_0^2 a_+^\dagger a_-^\dagger + (a_0^\dagger)^2 a_+ a_-.$$  
  Tunneling from mode 0 to the mode $+1$ and $-1$.
- In the particle picture
  $$|00\rangle \rightarrow \frac{1}{\sqrt{2}}(|1, -1\rangle + |-1, 1\rangle).$$
After some time, we have a state

\[ |n_0, n_{-1}, n_{+1}\rangle = |N - 2n, n, n\rangle. \]

Equivalently, \( N - 2n \) particles remained in the 0 state, while \( 2n \) particles form a symmetric Dicke state.
Experiment in the group of Carsten Klempt at the University of Hannover III

- Important: first excited spatial mode of the trap was used, not the ground state mode.

- It has two "bumps" rather than one, hence they had a split Dicke state.

Very simple entanglement criterion for singlets

- For separable states

\[
[\Delta(J^{(a)}_x + J^{(b)}_x)]^2 + [\Delta(J^{(a)}_y + J^{(b)}_y)]^2 + [\Delta(J^{(a)}_z + J^{(b)}_z)]^2 \geq \frac{N}{2}.
\]

For singlets, the LHS is zero.

- **Proof.** For product states \(|\psi_a\rangle \otimes |\psi_b\rangle\)

\[
\sum_{m=x,y,z} [\Delta(J^{(a)}_m + J^{(b)}_m)]^2 = \sum_{m=x,y,z} (\Delta J^{(a)}_m)^2 + \sum_{m=x,y,z} (\Delta J^{(a)}_m)^2 \geq \frac{N_a}{2} + \frac{N_b}{2}.
\]

holds.

- True also for separable states due to the concavity of the variance.
  
Very simple entanglement criterion for Dicke states

For separable states of two large spins

\[ \Delta(J_x^{(a)} - J_x^{(b)})^2 + \Delta(J_y^{(a)} - J_y^{(b)})^2 + \Delta(J_z^{(a)} + J_z^{(b)})^2 \geq \frac{N}{2}. \]

For Dicke states, the LHS is around \( \frac{N}{4} \) for large \( N \), since

\[ \Delta(J_z^{(a)} + J_z^{(b)})^2 = 0, \]
\[ \Delta(J_m^{(a)} + J_m^{(b)})^2 = \text{large}, \]
\[ \Delta(J_m^{(a)} - J_m^{(b)})^2 \approx \frac{N}{8} = \text{small} \]

for \( m = x, y \).

Not a practical criterion since small noise makes the state undetectable, and it assumes symmetry.
Number-phase-like uncertainty

- We start from the sum of two Heisenberg uncertainty relations

\[
(\Delta J_z)^2[(\Delta J_x)^2 + (\Delta J_y)^2] \geq \frac{1}{4} (\langle J_x \rangle^2 + \langle J_y \rangle^2).
\]

Then,

\[
(\Delta J_z)^2[(\Delta J_x)^2 + (\Delta J_y)^2] + \frac{1}{4}[(\Delta J_x)^2 + (\Delta J_y)^2] \geq \frac{1}{4} (\langle J_x^2 \rangle + \langle J_y^2 \rangle).
\]

- Simple algebra yields

\[
\left[ (\Delta J_z)^2 + \frac{1}{4} \right] \times \frac{(\Delta J_x)^2 + (\Delta J_y)^2}{\langle J_x^2 \rangle + \langle J_y^2 \rangle} \geq \frac{1}{4}.
\]

- Note that \(\langle J_x^2 \rangle\) appears, not \(\langle J_x \rangle^2\).
Let us introduce the normalized variables

$$\tilde{J}_m^{(n)} = \frac{J_m^{(n)}}{J_n^{(n)}},$$

where $m = x, y$ and $n = a, b$ (i.e., left well, right well), the total spin is

$$J_n = \frac{N_n}{2},$$

and

$$\mathcal{J}^{(n)} = \left( \frac{(J_x^{(n)})^2 + (J_y^{(n)})^2}{j_n^2} \right)^{\frac{1}{2}}.$$

$\mathcal{J}^{(n)} \approx 1$ indicates a state close to being symmetric in the well, which is the case ideally. In general, $\mathcal{J}^{(n)} \leq 1$. 
Our uncertainty relation is now

$$\left[ (\Delta J_z)^2 + \frac{1}{4} \right] \left[ (\Delta \tilde{J}_x)^2 + (\Delta \tilde{J}_y)^2 \right] \geq \frac{1}{4}.$$ 

We define

$$J_z^+ = J_z^{(a)} + J_z^{(b)},$$
$$\tilde{J}_m^- = \tilde{J}_m^{(a)} - \tilde{J}_m^{(b)}$$

for $m = x, y$. 

Uncertainty with normalized variables
The two-well entanglement criterion

Suggestion of the experimentalists: we need a product criterion, since it is good for realistic noise.

Main result

For separable states,

\[
\left[ (\Delta J_z^+)^2 + \frac{1}{2} \right] \times \left[ \langle (\tilde{J}_x^-)^2 + (\tilde{J}_y^-)^2 \rangle \right] \geq f(J^{(a)}, J^{(b)})
\]

holds, where \( f(x, y) = \frac{(x^2 + y^2 - 1)^2}{xy} \),

Any state violating the inequality is entangled.
For $j_a = j_b = \frac{N}{4}$ and symmetric states in the wells

\[
\left[(\Delta J_z)^2 + \frac{1}{2}\right]\left[(\Delta (J_x^{(a)} - J_x^{(b)}))^2 + (\Delta (J_y^{(a)} - J_y^{(b)}))^2\right] \geq \frac{N^2}{16}.
\]
Problem 1: Varying particle number

- The experiment is repeated many times. Each time we find a somewhat different particle number.

- Postselecting for a given particle number is not feasible.

- Consider a density matrix

\[ \rho = \sum_{j_a, j_b} Q_{j_a, j_b} \rho_{j_a, j_b}, \]

where \( \rho_{j_a, j_b} \) are states with \( 2j_a \) and \( 2j_b \) particles in the two wells, \( Q_{j_a, j_b} \) are probabilities.

- \( \rho \) is entangled iff at least one of the \( \rho_{j_a, j_b} \) is entangled.

- We use special normalization in the criterion.
Problem 2: States are not always symmetric in a BEC of two-state atoms

- Ideally, the BEC is in a single spatial mode.

- The state of an ensemble of the two-state atoms must be symmetric.

- In practice, the BEC is not in a single spatial mode, so there is no perfect symmetry.

- Our criterion must handle this.
Correlations for Dicke states

For the Dicke state

\[(\Delta (J_x^{(a)} - J_x^{(b)}))^2 \approx 0,\]
\[(\Delta (J_y^{(a)} - J_y^{(b)}))^2 \approx 0,\]
\[(\Delta J_z)^2 = 0.\]

Measurement results on well "b" can be predicted from measurements on "a"

\[J_x^{(b)} \approx J_x^{(a)},\]
\[J_y^{(b)} \approx J_y^{(a)},\]
\[J_z^{(b)} = -J_z^{(a)}.\]
Here, $J_{\perp}^{(n)} = \cos \alpha J_x^{(n)} + \sin \alpha J_y^{(n)}$. 
Further experimental results

A. Black: shot-noise limit. Green circles: experiments.
C: Black: perfect symmetry. Blue/red: values for the left/right well.

\[ (\Delta J_i^2) = \sqrt{ \left\langle (J_i^{(n)})^2 + (J_i^{(n)})^2 \right\rangle} \]
Violation of the criterion: entanglement is detected

\[ \left( \frac{\Delta J_z^+}{2} \right) \times \left( \frac{1}{2} \right) \geq f \left( \mathcal{J}^{(a)}, \mathcal{J}^{(b)} \right) \]

For separable states,

\[
\left[ (\Delta J_z^+)^2 + \frac{1}{2} \right] \times \left[ \langle \tilde{J}_x^- \rangle^2 + \langle \tilde{J}_y^- \rangle^2 \right] \geq f \left( \mathcal{J}^{(a)}, \mathcal{J}^{(b)} \right)
\]

holds, where \( f(x, y) = \frac{(x^2 + y^2 - 1)^2}{xy} \).
Collaborators on entanglement conditions for Dicke states

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Detection of bipartite entanglement close to Dicke states.
Non-symmetric states within the wells and a varying particle number can also be handled.


THANK YOU FOR YOUR ATTENTION! FOR TRANSPARENCIES, PLEASE SEE www.gtoth.eu.
Appendix
Proof

*Product states.* For states of the form \(|\psi^{(a)}\rangle \otimes |\psi^{(b)}\rangle\).

\[
\left[ (\Delta J^+_z)^2 + \frac{1}{2} \right] \times \left[ (\Delta \tilde{J}^-_x)^2 + (\Delta \tilde{J}^-_y)^2 \right] \\
= \left[ (U^{(a)} + \frac{1}{4}) + (U^{(b)} + \frac{1}{4}) \right] \cdot (V^{(a)} + V^{(b)}) \\
\geq 4 \sqrt{(U^{(a)} + \frac{1}{4})(U^{(b)} + \frac{1}{4})V^{(a)}V^{(b)}} \geq 1
\]

holds, where we used the notation

\[
U^{(n)} = (\Delta J^{(n)}_z)^2, \quad V^{(n)} = (\Delta \tilde{J}^{(n)}_x)^2 + (\Delta \tilde{J}^{(n)}_y)^2
\]

for \(n = a, b\). We used that

(i) \([\Delta (A^{(a)} + A^{(b)})]^2 = (\Delta A^{(a)})^2 + (\Delta A^{(b)})^2\),

(ii) Inequality between the arithmetic and the geometric mean,

(iii) Our number-phase like uncertainty.
Proof II

Using $\langle (\tilde{J}_x^{(n)})^2 \rangle + \langle (\tilde{J}_y^{(n)})^2 \rangle = 1$ for $n = a, b$, our inequality for product states yields

$$2 \left[ (\Delta J_z^+)^2 + \frac{1}{2} \right] (S - C) \geq S,$$

where correlations between the two subsystems are characterized by

$$C = \left\langle \frac{J_x^{(a)} J_x^{(b)} + J_y^{(a)} J_y^{(b)}}{j_a j_b} \right\rangle,$$

and

$$S = J^{(a)} J^{(b)}.$$

$C$ can be negative and $|C| \leq S$.

The normalization with the total spin will make it easier to adapt our criterion to experiments with a varying particle number in the ensembles.
Separable states. We now consider a mixed separable state of the form \( \varrho_{\text{sep}} = \sum_k p_k |\psi_k^{(a)}\rangle \otimes |\psi_k^{(b)}\rangle \). For such states, we can write the following series of inequalities

\[
2 \left[ (\Delta J^+_z)^2 + \frac{1}{2} \right] (S - C) \geq 2 \left[ \sum_k p_k (\Delta J_z^2)_k + \frac{1}{2} \right] \left[ \sum_k p_k (S_k - C_k) \right] \\
\geq 2 \sum_k p_k \sqrt{\left( (\Delta J_z^2)_k + \frac{1}{2} \right) (S_k - C_k)} \geq \left( \sum_k p_k \sqrt{S_k} \right)^2,
\]

Subscript \( k \) refers to the \( k^{\text{th}} \) sub-ensemble \( |\psi_k^{(a)}\rangle \otimes |\psi_k^{(b)}\rangle \).

(i) The first inequality in is due to \( (\Delta J^+_z)^2 \) and \( S \) being concave in the quantum state.

(ii) The second inequality is based on the Cauchy-Schwarz inequality.

(iii) The third inequality is the application of the previous inequality for all sub-ensembles.
Next, we find a lower bound on the RHS of the last inequality based on the knowledge of $J^{(a)}$ and $J^{(b)}$. We find that

$$\sum_{k} \rho_k \left( J^{(a)}_k J^{(b)}_k \right)^{1/2} \geq (J^{(a)})^2 + (J^{(b)})^2 - 1,$$

which is based on noting $(xy)^{1/4} \geq x + y - 1$ for $0 \leq x, y \leq 1$.

Using this to bound the RHS from below and dividing by $S$ we obtain

$$\left[ (\Delta J^+_z)^2 + \frac{1}{2} \right] \times \left[ 2 - 2\frac{C}{S} \right] \geq \frac{\left( (J^{(a)})^2 + (J^{(b)})^2 - 1 \right)^2}{S}.$$