

Quantum states with a positive partial transpose are useful for metrology

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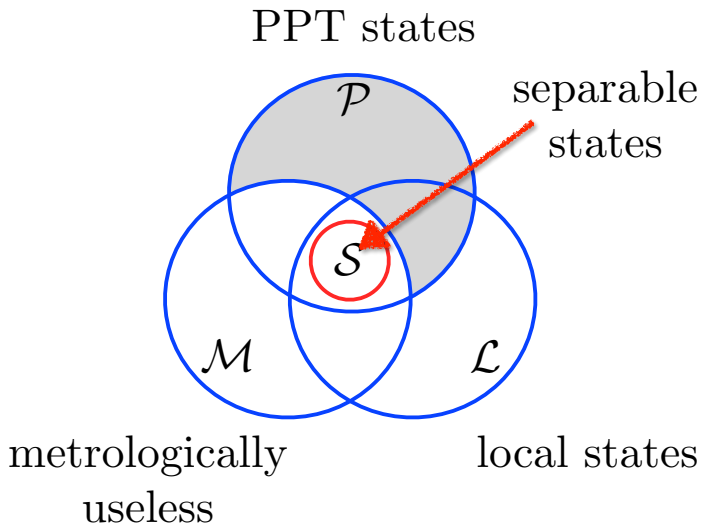
20 May 2018

- 1 Motivation**
 - What are entangled states useful for?
- 2 Background**
 - Quantum Fisher information
- 3 Maximizing the QFI for PPT states**
 - Results so far
 - Our results

What are entangled states useful for?

- Entangled states are useful, but not all of them are useful for some task.
- Entanglement is needed for beating the shot-noise (=classical) limit in quantum metrology.
- Intriguing question: Are states with a positive partial transpose useful for metrology? Can they also beat the shot-noise limit?

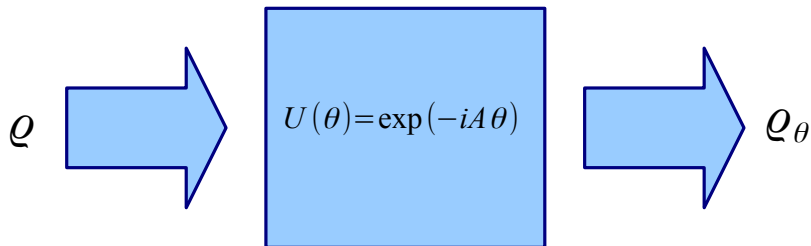
What are entangled states useful for?



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Quantum metrology

- Fundamental task in metrology



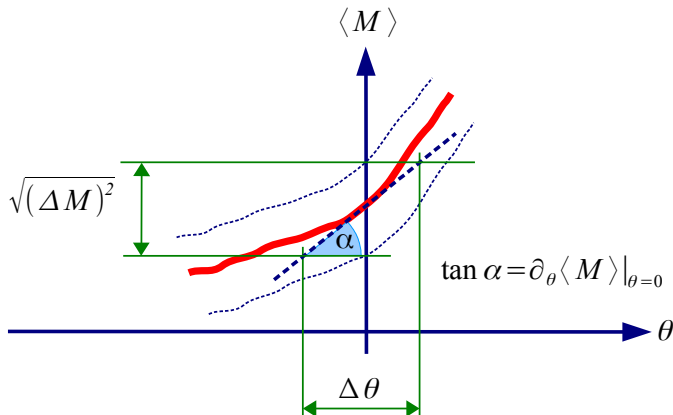
- We have to estimate θ in the dynamics

$$U = \exp(-iA\theta).$$

Precision of parameter estimation

- Measure an operator M to get the estimate θ . The precision is

$$(\Delta\theta)^2 = \frac{(\Delta M)^2}{|\partial_\theta \langle M \rangle|^2}.$$



The quantum Fisher information

Cramér-Rao bound on the precision of parameter estimation

$$(\Delta\theta)^2 \geq \frac{1}{F_Q[\varrho, A]}, \quad \frac{1}{(\Delta\theta)^2} \leq F_Q[\varrho, A].$$

where $F_Q[\varrho, A]$ is the **quantum Fisher information**.

- The quantum Fisher information is

$$F_Q[\varrho, A] = 2 \sum_{k,l} \frac{(\lambda_k - \lambda_l)^2}{\lambda_k + \lambda_l} |\langle k|A|l \rangle|^2,$$

where $\varrho = \sum_k \lambda_k |k\rangle\langle k|$.

- Linear interferometer: collective operators $A \in \{J_x, J_y, J_z\}$.

The quantum Fisher information vs. entanglement

- **Shot-noise limit:** For N -qubit separable states

$$F_Q[\varrho, J_l] \leq N, \quad l = x, y, z.$$

[Pezze, Smerzi, Phys. Rev. Lett. 102, 100401 (2009);

Hyllus, GÃCEhne, Smerzi, Phys. Rev. A 82, 012337 (2010)]

- **A quantum state is "useful" if it violates the above inequality.**
- **Heisenberg limit:** For entangled states

$$F_Q[\varrho, J_l] \leq N^2, \quad l = x, y, z.$$

where the bound can be saturated (GHZ states).

Outline

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States with a Positive-semidefinite Partial Transpose (PPT)

A density matrix of a separable state fulfils

$$\rho^{T1} \geq 0,$$

where **T1 denotes partial transpose.**

- Any state violating the inequality is entangled.
- Bound entangled: entangled, but still PPT.

[A. Peres,PRL 77, 1413 (1996);

M. Horodecki, P. Horodecki, and R. Horodecki,PLA 223, 1 (1996).]

Results so far concerning metrologically useful PPT states

- Bound entangled states with PPT and some non-PPT partitions.
- Violates an entanglement criterion with three QFI terms.
[P. Hyllus, W. Laskowski, R. Krischek, C. Schwemmer, W. Wieczorek, H. Weinfurter, L. Pezze, and A. Smerzi, PRA 85, 022321 (2012).]
- Non-unlockable bound entangled states with PPT and some non-PPT partitions.
- Violates the criterion with a single QFI term, better than shot-noise limit.
[Ł. Czekaj, A. Przysiężna, M. Horodecki, P. Horodecki, Phys. Rev. A 92, 062303 (2015).]

on nonlocality [43]) to answer would be, Is there any family of quantum states that allows for a general Local Hidden Variables (LHV) model but can be used to obtain sub-shot-noise (i.e., better than classical) quantum metrology? This question is related to another question (especially in the context of both general requirements in quantum metrology [26] and recent results on nonlocality [43]) regarding whether there is any chance for sub-shot-noise metrology for states obeying the PPT condition with respect to *any* cut. While the present result

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Our results

We look for

- bipartite PPT entangled states
- multipartite states that are PPT with respect to **all** partitions.

Maximizing the QFI for PPT states: brute force

Maximize the QFI for PPT states. Remember

$$F_Q[\varrho, A] = 2 \sum_{k,l} \frac{(\lambda_k - \lambda_l)^2}{\lambda_k + \lambda_l} |\langle k|A|l \rangle|^2,$$

where $\varrho = \sum_k \lambda_k |k\rangle\langle k|$.

- Difficult to maximize a convex function over a convex set. The maximum is taken on the boundary of the set.
- Not guaranteed to find the global maximum.
- Note: Finding the *minimum* is possible!

Maximizing the QFI for PPT state: our method

- We mentioned that the QFI gives a bound on the precision of the parameter estimation

$$F_Q[\varrho, A] \geq \frac{1}{(\Delta\theta)_M^2} = \frac{|\partial_\theta \langle M \rangle|^2}{(\Delta M)^2} = \frac{\langle i[M, A] \rangle^2}{(\Delta M)^2} \quad (\text{dynamics is } U = e^{-iA\theta})$$

- The bound is sharp

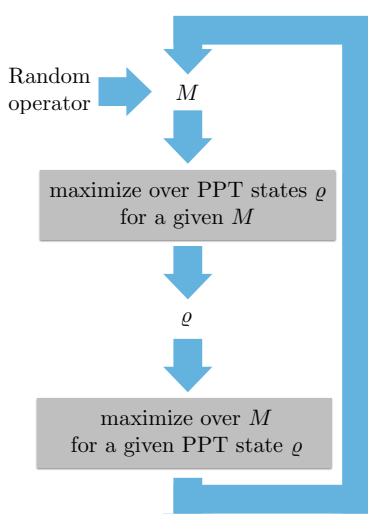
$$F_Q[\varrho, A] = \max_M \frac{1}{(\Delta\theta)_M^2}.$$

[M. G. Paris, *Int. J. Quantum Inform.* 2009. Used, e.g., in F. Fröwis, R. Schmied, and N. Gisin, 2015; I. Appelaniz *et al.*, *NJP* 2015.]

The maximum for PPT states can be obtained as

$$\max_{\varrho \text{ is PPT}} F_Q[\varrho, A] = \max_{\varrho \text{ is PPT}} \max_M \frac{\langle i[M, A] \rangle_\varrho^2}{(\Delta M)^2}.$$

Saw-saw algorithm for maximizing the precision



See also K. Macieszczak, [arXiv:1312.1356v1](https://arxiv.org/abs/1312.1356v1) for an iterative algorithm for optimizing over noisy states.

Maximize over PPT states ρ for a given M

Best precision for PPT states for a given operator M ,
semidefinite program:

Proof.—Let us define first

$$\begin{aligned} f_M(X, Y) = \min_{\rho} \quad & \text{Tr}(M^2 \rho), \\ \text{s.t.} \quad & \rho \geq 0, \rho^{\text{Tk}} \geq 0 \text{ for all } k, \text{Tr}(\rho) = 1, \\ & \langle i[M, A] \rangle = X \text{ and } \langle M \rangle = Y. \end{aligned}$$

The best precision for a given M and for PPT states is

$$(\Delta\theta)^2 = \min_{X, Y} \frac{f_M(X, Y) - Y^2}{X^2}.$$

The state giving the best precision is ρ_{PPTopt} .

Maximize over M for a given PPT state ϱ

For a state ϱ , the best precision is obtained with the operator given by the symmetric logarithmic derivative

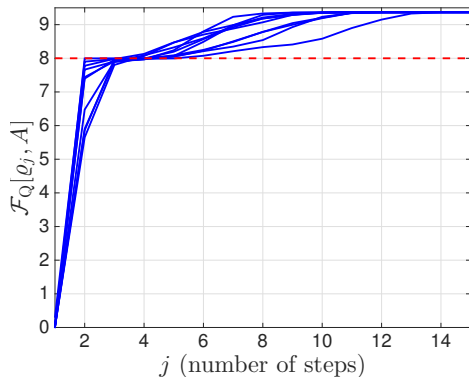
$$M = 2i \sum_{k,l} \frac{\lambda_k - \lambda_l}{\lambda_k + \lambda_l} |k\rangle \langle l| \langle k|A|l\rangle,$$

where $\varrho = \sum_k \lambda_k |k\rangle \langle k|$.

Convergence of the method

The precision cannot get worse with the iteration!

Convergence of the method II



Generation of the 4×4 bound entangled state.

- (blue) 10 attempts. After 15 steps, the algorithm converged.
- (red) Maximal quantum Fisher information for separable states.

Robustness of the states

$$\varrho(p) = (1 - p)\varrho + p\varrho_{\text{noise}}$$

- Robustness of entanglement: the maximal p for which $\varrho(p)$ is entangled for any separable ϱ_{noise} .
[Vidal and Tarrach, PRA 59, 141 (1999).]
- **Robustness of metrological usefulness**: the maximal p for which $\varrho(p)$ outperforms separable state for any separable ϱ_{noise} .

Robustness of the states II

System	A	$\mathcal{F}_Q[\varrho, A]$	$\mathcal{F}_Q^{(\text{sep})}$	$\rho_{\text{white noise}}$
four qubits	J_z	4.0088	4	0.0011
three qubits	$j_z^{(1)} + j_z^{(2)}$	2.0021	2	0.0005
2×4 (three qubits, only 1 : 23 is PPT)	$j_z^{(1)} + j_z^{(2)}$	2.0033	2	0.0008

Multiqubit states

Robustness of the states III

d	$\mathcal{F}_Q[\varrho, A]$	$\rho_{\text{white noise}}$	$\rho_{\text{noise}}^{\text{LB}}$
3	8.0085	0.0006	0.0003
4	9.3726	0.0817	0.0382
5	9.3764	0.0960	0.0361
6	10.1436	0.1236	0.0560
7	10.1455	0.1377	0.0086
8	10.6667	0.1504	0.0670
9	10.6675	0.1631	0.0367
10	11.0557	0.1695	0.0747
11	11.0563	0.1807	0.0065
12	11.3616	0.1840	0.0808

- $d \times d$ systems.
- Maximum of the quantum Fisher information for separable states is 8.
- The operator A is not the usual J_z .

4×4 bound entangled PPT state

Let us define the following six states

$$|\Psi_1\rangle = (|0, 1\rangle + |2, 3\rangle)/\sqrt{2}, \quad |\Psi_2\rangle = (|1, 0\rangle + |3, 2\rangle)/\sqrt{2},$$

$$|\Psi_3\rangle = (|1, 1\rangle + |2, 2\rangle)/\sqrt{2}, \quad |\Psi_4\rangle = (|0, 0\rangle - |3, 3\rangle)/\sqrt{2},$$

$$|\Psi_5\rangle = (1/2)(|0, 3\rangle + |1, 2\rangle) + |2, 1\rangle/\sqrt{2},$$

$$|\Psi_6\rangle = (1/2)(-|0, 3\rangle + |1, 2\rangle) + |3, 0\rangle/\sqrt{2}.$$

Our state is a mixture

$$\rho_{4 \times 4} = p \sum_{n=1}^4 |\Psi_n\rangle\langle\Psi_n| + q \sum_{n=5}^6 |\Psi_n\rangle\langle\Psi_n|,$$

where $q = (\sqrt{2} - 1)/2$ and $p = (1 - 2q)/4$. We consider the operator

$$A = H \otimes \mathbb{1} + \mathbb{1} \otimes H,$$

where $H = \text{diag}(1, 1, -1, -1)$.

Negativity

Apart from making calculations for PPT bound entangled states, we can also make calculations for states with a given negativity.

[G. Vidal and R. F. Werner, PRA 65, 032314 (2002).]

Entanglement

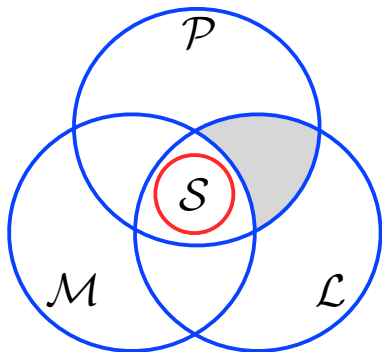
Bipartite state	Entanglement
3×3	0.0003
4×4	0.0147
5×5	0.0239
6×6	0.0359
7×7	0.0785
UPB 3×3	0.0652
Breuer 4×4	0.1150

Convex roof of the linear entanglement entropy. The entanglement is also shown for the 3×3 state based on unextendible product bases (UPB) and for the Breuer state with a parameter $\lambda = 1/6$.

[G. Tóth, T. Moroder, and O. Gühne, PRL 114, 160501 (2015).]

PPT, metrologically useful, not violating any Bell inequality

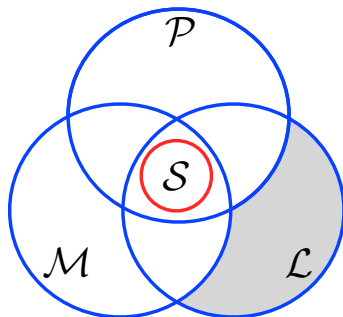
We construct an LHV model for the 2×4 state listed before.



[F. Hirsch, M. T. Quintino, T. Vértesi, M. F. Pusey, and N. Brunner, PRL 2016;
D. Cavalcanti, L. Guerini, R. Rabelo, and P. Skrzypczyk, PRL 2016.]

Non-PPT, metrologically useful, not violating any Bell inequality

- Two-qubit Werner state $p|\Psi^-\rangle\langle\Psi^-| + (1-p)\mathbb{1}/4$, with $|\Psi^-\rangle = (|01\rangle - |10\rangle)/\sqrt{2}$.
- Better for metrology than separable states ($\mathcal{F}_Q > 2$) for $p > 1 - 0.3596 = 0.6404$.
- They do not violate a Bell inequality for $p < 0.6829$.



Cluster states: non-PPT, not useful, violates a Bell inequality

Cluster states: resource in measurement-based quantum computing

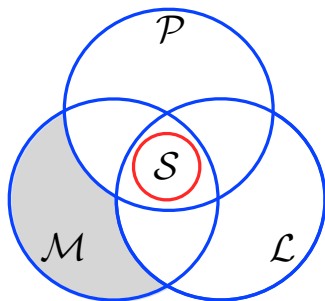
[R. Raussendorf and H. J. Briegel, PRL 2001.]

- Fully entangled pure states.
- Violate a Bell inequality

[V. Scarani, A. Acín, E. Schenck, M. Aspelmeyer, PRA 2005; O. Gühne, GT, P. Hyllus, H. J. Briegel, PRL 2005; GT, O. Gühne, and H. J. Briegel, PRA 2006.]

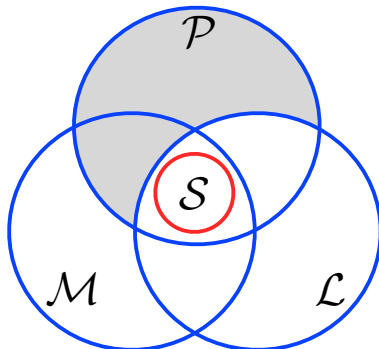
- Certain cluster states are metrologically not useful

[P. Hyllus, O. Gühne, and A. Smerzi, PRA 2010.]



PPT, violates a Bell inequality

Counterexample for the Peres conjecture



[T. Vértesi and N. Brunner, Nature Communications 2015.]

Summary

- We presented quantum states with a positive partial transpose with respect to all bipartitions that are useful for metrology.

See:

Géza Tóth and Tamás Vértesi,

Quantum states with a positive partial transpose
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Phys. Rev. Lett. 120, 020506 (2018).

<http://gtoth.eu>

THANK YOU FOR YOUR ATTENTION!

