Activating hidden metrological usefulness

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1 Motivation
   - What are entangled states useful for?

2 Background
   - Quantum Fisher information
   - Recent findings on the quantum Fisher information

3 Metrological gain and the optimal local Hamiltonian
   - Metrological usefulness of a quantum state.
   - Activation of metrological usefulness
   - Optimal local Hamiltonian
   - All bipartite pure entangled states are useful
What are entangled states useful for?

- Entangled states are useful, but not all of them are useful for some task.

- Entanglement is needed for beating the shot-noise limit in quantum metrology.

Intriguing questions:

- Can a quantum state become useful metrologically, if an ancilla or a second copy is added?

- How to find the local Hamiltonian, for which a quantum state is the most useful compared to separable states?
What are entangled states useful for?

- PPT states
- Separable states
- Metrologically useless
- Local states

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Quantum metrology

**Fundamental task in metrology**

\[ U(\theta) = \exp(-iA\theta) \]

We have to estimate \( \theta \) in the dynamics

\[ U = \exp(-iA\theta). \]
Cramér-Rao bound on the precision of parameter estimation

\[(\Delta \theta)^2 \geq \frac{1}{mF_Q[\varrho, A]},\]

where \(m\) is the number of independent repetitions and \(F_Q[\varrho, A]\) is the quantum Fisher information.

The quantum Fisher information is

\[F_Q[\varrho, A] = 2 \sum_{k,l} \frac{(\lambda_k - \lambda_l)^2}{\lambda_k + \lambda_l} |\langle k|A|l \rangle|^2,\]

where \(\varrho = \sum_k \lambda_k |k\rangle \langle k|\).
Measure an operator $M$ to get the estimate $\theta$. The error propagation formula is

$$(\Delta \theta)^2_M = \frac{(\Delta M)^2}{|\partial_\theta \langle M \rangle|^2}.$$
Relation between $\left(\Delta \theta\right)^2$ and the error propagation formula $\left(\Delta \theta\right)^2_M$

- The relation

$$\left(\Delta \theta\right)^2 \geq \frac{1}{m} \left(\Delta \theta\right)^2_{M_{\text{opt}}}$$

holds, where $m$ is the number of independent repetitions and $M_{\text{opt}}$ is the optimal observable.

- The relation can be saturated if $m$ is large and the distribution fulfills certain requirements.


- Moreover,

$$\left(\Delta \theta\right)^2_{M_{\text{opt}}} = \frac{1}{F_Q[\rho, A]}.$$

Special case $A = J_l$

- The operator $A$ is defined as

$$A = J_l = \sum_{n=1}^{N} j_l^{(n)}, \quad l \in \{x, y, z\}.$$ 

- Magnetometry with a linear interferometer
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Many bounds on the quantum Fisher information can be derived from these simple properties:

- For pure states, it equals four times the variance,

  \[ F[|\psi\rangle\langle\psi|, A] = 4(\Delta A)^2_\psi. \]

- For mixed states, it is convex.
The quantum Fisher information vs. entanglement

- For separable states

\[ F_Q[\varrho, J_l] \leq N, \quad l = x, y, z. \]


- For states with at most \( k \)-particle entanglement (\( k \) is divisor of \( N \))

\[ F_Q[\varrho, J_l] \leq kN. \]


- Macroscopic superpositions (e.g, GHZ states, Dicke states)

\[ F_Q[\varrho, J_l] \propto N^2, \]

Even a little uncorrelated local noise leads to shot-noise scaling above a certain particle number.

The quantum Fisher information is the convex roof over the purifications of the dynamics

$$F_Q[\rho_\theta] = \min_{|\psi_\theta\rangle} F_Q[|\psi_\theta\rangle],$$

where the purification is $|\psi_\theta\rangle$ is related to the state as

$$\rho_\theta = \text{Tr}_E(|\psi_\theta\rangle\langle\psi_\theta|).$$

Most important characteristics used for estimation

The quantum Fisher information is the convex roof of the variance

\[ F_Q[\varrho, \mathcal{A}] = 4 \min_{\rho_k, \Psi_k} \sum_k \rho_k (\Delta \mathcal{A})^2_k, \]

where

\[ \varrho = \sum_k \rho_k |\Psi_k\rangle \langle \Psi_k|. \]


- Extended convexity for non-unitary dynamics.
Witnessing the quantum Fisher information based on few measurements

- Let us bound the quantum Fisher information based on some measurements.

\[ F_Q = N^2(1 - 2F_{\text{GHZ}})^2 \]

if \( F_{\text{GHZ}} > \frac{1}{2} \)

Quantum Fisher information vs. Fidelity with respect to (a) GHZ states and (b) Dicke states for \( N = 4, 6, 12 \).

Continuity of QFI and QFI for symmetric states

- Arbitrarily small entanglement can be used to get close to Heisenberg scaling.
- The difference between the QFI of two states can be bounded by the distance of the two states.
- Bound on the QFI with the geometric measure of entanglement.

[R. Augusiak, J. Kołodyński, A. Streltsov, M. N. Bera, A. Acín, M. Lewenstein, PRA 2016]

- Continuity in the non-unitary case:

[A. T. Rezakhani, S. Alipour, M. Hassani, PRA 2019]

- Random pure states of distinguishable particles typically do not lead to super-classical scaling of precision.
- Random states from the symmetric subspace typically achieve the optimal Heisenberg scaling.

[M. Oszmaniec, R. Augusiak, C. Gogolin, J. Kołodyński, A. Acín, M. Lewenstein, PRX 2016]
Metrologically useful PPT states

- Bound entangled states with PPT and some non-PPT partitions.
  
  Violates an entanglement criterion with three QFI terms.
  

- Non-unlockable bound entangled states with PPT and some non-PPT partitions.
  
  Violates the criterion with a single QFI term, better than shot-noise limit.
  

- Bipartite PPT entangled states can be useful for metrology, and they are close to be maximally useful for large dimensions.
  
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Metrological usefulness

- Metrological gain for a given Hamiltonian

\[ g_{\mathcal{H}}(\varrho) = \frac{\mathcal{F}_Q[\varrho, \mathcal{H}]}{\mathcal{F}_Q^{(\text{sep})}(\mathcal{H})}, \]

where \( \mathcal{F}_Q^{(\text{sep})}(\mathcal{H}) \) is the maximum of the QFI for separable states.

- Metrological gain optimized over all local Hamiltonians

\[ g(\varrho) = \max_{\text{local } \mathcal{H}} g_{\mathcal{H}}(\varrho) = \max_{\text{local } \mathcal{H}} \frac{\mathcal{F}_Q[\varrho, \mathcal{H}]}{\mathcal{F}_Q^{(\text{sep})}(\mathcal{H})}. \]

- The metrological gain is convex in the state.

  [G. Toth, T. Vertesi, P. Horodecki, R. Horodecki, PRL 2020.]

- We would like to determine \( g \).
So we would like optimize over local $\mathcal{H}$ the expression

$$g(\varrho) = \max_{\text{local} \mathcal{H}} \frac{\mathcal{F}_Q[\varrho, \mathcal{H}]}{\mathcal{F}_Q^{(\text{sep})}(\mathcal{H})}.$$ 

First observation: we really optimize the QFI over $\mathcal{H}$, but we normalize it with something meaningful.

This is needed, since otherwise $\mathcal{H}' = 100\mathcal{H}$ would be better than $\mathcal{H}$.

Second observation: difficult task, since both the numerator and the denominator depend on $\mathcal{H}$. 
Maximally entangled state

- It is a difficult task to obtain $g(\varrho)$ and the optimal local Hamiltonian for any $\varrho$.

- As a first step, we consider the $d \times d$ maximally entangled state, which is defined as

$$|\psi^{(\text{me})}\rangle = \frac{1}{\sqrt{d}} \sum_{k=1}^{d} |k\rangle |k\rangle.$$  

- Due to the symmetry of the state, the optimal Hamiltonian is

$$\mathcal{H}^{(\text{me})} = D \otimes 1 + 1 \otimes D,$$

where the diagonal matrix $D$ is given as

$$D = \text{diag}(+1, -1, +1, -1, ...).$$
For the $3 \times 3$-case, we consider the noisy quantum state

$$\rho_{AB}^{(p)} = (1 - p)|\psi^{(\text{me})}\rangle\langle\psi^{(\text{me})}| + p \mathbb{1}/d^2,$$

which is useful if

$$p < \frac{25 - \sqrt{177}}{32} \approx 0.3655.$$

Note that it is entangled if

$$p < \frac{2}{3}.$$
Maximally entangled state III

\[ H_{\text{coll}}(H^{(\text{iso})}) = H^{(\text{iso})} \otimes \mathbb{1} + \mathbb{1} \otimes (H^{(\text{iso})})^*, \]

(- - - -) Separable states.
(● ● ● ●) Isotropic states for two-body Hamiltonians \( H_{\text{coll}}(H^{(\text{iso})}) \), where \( H^{(\text{iso})} \) are chosen randomly.
(● ● ● ●) Isotropic states with a larger \( p \) are useful for metrology.
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(a) An ancilla ("a") is added to bipartite state $\rho_{AB}$.

(b) An additional copy or a different state is added to the state.
Activation by an ancilla qubit

- Now we consider the previous state, after a pure ancilla qubit is added

\[ \rho^{(\text{anc})} = |0\rangle\langle 0|_a \otimes \rho^{(p)}_{AB}. \]

- Then, with the operator

\[ \mathcal{H}^{(\text{anc})} = 1.2C_{aA} \otimes 1_B + 1_a \otimes D_B, \]

where an operator acting on the ancilla and A is

\[ C_{aA} = \frac{9}{20} (2\sigma_x + \sigma_z)_a \otimes |0\rangle\langle 0|_a + 1_a \otimes (|2\rangle\langle 2|_a - |1\rangle\langle 1|_a), \]

we have \( g_{\mathcal{H}^{(\text{anc})}}(\rho^{(\text{anc})}) > 1 \) if

\[ p < 0.3752. \]

- Hence larger part of the noisy maximally entangled states are useful in the case with the ancilla (For a single copy, the limit was \( p < 0.3655 \).)
Activation by a second copy

We consider now two copies of the noisy $3 \times 3$ maximally entangled state

$$\varrho^{(tc)} = \varrho_{AB}^{(p)} \otimes \varrho_{A'B'}^{(p)}.$$ 

Then, with the two-copy operator

$$\mathcal{H}^{(tc)} = D_A \otimes D_{A'} \otimes 1_{BB'} + 1_{AA'} \otimes D_{B} \otimes D_{B'},$$

we have $g_{\mathcal{H}(tc)}(\varrho^{(tc)}) > 1$ if

$$p < 0.4164.$$ 

Hence larger part of the noisy maximally entangled states are useful in the two-copy case, than with a single copy. (For a single copy, the limit was $p < 0.3655.$)
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For qubits, the local Hamiltonians with eigenvalues $+1$ and $-1$ differ from each other by local unitaries

$$
\mathcal{H} = U_1 \sigma_z U_1^\dagger \otimes 1 + 1 \otimes U_2 \sigma_z U_2^\dagger.
$$

It is possible to obtain bounds on the quantum Fisher information.

All pure two-qubit entangled states are useful, while not all pure multi-qubit entangled states are useful.


When looking at $\mathcal{F}_Q / \mathcal{F}_Q^{(sep)}$, the value of $\mathcal{F}_Q^{(sep)}$ does not depend on the particular Hamiltonian. For instance for spin operators $\mathcal{F}_Q^{(sep)} = N$.

Method for finding the optimal local Hamiltonian for qudits with \( d > 2 \)

- The case of qudits is more complicated than the case of qubits, since the local Hamiltonians cannot be converted to each other by unitaries.

- We need to maximize \( \mathcal{F}_Q[\varrho, \mathcal{H}] \) over \( \mathcal{H} \) for a given \( \varrho \). However, \( \mathcal{F}_Q \) convex in \( \mathcal{H} \), maximizing it over \( \mathcal{H} \) is a difficult task.

- Instead of the quantum Fisher information, let us consider the error propagation formula

\[
(\Delta \theta)^2_M = \frac{(\Delta M)^2}{\langle i[M, \mathcal{H}] \rangle^2},
\]

which provides a bound on the quantum Fisher information

\[
\mathcal{F}_Q[\varrho, \mathcal{H}] \geq 1/(\Delta \theta)^2_M.
\]

We will now minimize \( (\Delta \theta)^2_M \).
To be more specific, we use

$$\mathcal{F}_Q[\varrho, \mathcal{H}] = \max_M \frac{\langle i[M, \mathcal{H}] \rangle^2}{\varrho^2 (\Delta M)^2}.$$ 

The maximum over local Hamiltonians can be obtained as

$$\max_{\text{local } \mathcal{H}} \mathcal{F}_Q[\varrho, \mathcal{H}] = \max_{\text{local } \mathcal{H}} \max_M \frac{\langle i[M, \mathcal{H}] \rangle^2}{(\Delta M)^2}.$$ 

Similar idea for optimizing over the state, rather than over $\mathcal{H}$:

See-saw algorithm for maximizing the precision for given $c_1, c_2$

Note that $\mathcal{H}_1, \mathcal{H}_2$ fulfill

$$c_n \mathbb{1} \pm \mathcal{H}_n \geq 0.$$
Maximize over $\mathcal{H}$

- We have to maximize
  \[ \langle i[M, \mathcal{H}] \rangle. \]

- Simple algebra yields
  \[ \langle i[M, \mathcal{H}] \rangle = \text{Tr}(A_1 \mathcal{H}_1) + \text{Tr}(A_2 \mathcal{H}_2), \]
  where
  \[ A_n = \text{Tr}_{\{1,2\}\setminus n}(i[q, M]) \]
  are operators acting on a single subsystem.

- Hence, we can maximize $\langle i[M, \mathcal{H}] \rangle$ over $\mathcal{H}_1$ and $\mathcal{H}_2$. 
Maximize over $\mathcal{H}$ II

- The optimal $\mathcal{H}_n$ is the one that maximizes $\text{Tr}(A_n\mathcal{H}_n)$ under these constraints. It can straightforwardly be obtained as

$$\mathcal{H}^{(\text{opt})}_n = U_n \tilde{D}_n U_n^\dagger,$$

where the eigendecomposition of $A$ is given as

$$A_n = U_n D_n U_n^\dagger$$

and

$$(\tilde{D}_n)_{k,k} = c_n s((D_n)_{k,k}),$$

where $s(x) = 1$ if $x \geq 0$, and $-1$ otherwise.

- Clearly, $\mathcal{H}^{(\text{opt})}_n$ has the same eigenvectors as $A_n$ and has only eigenvalues $+c_n$ and $-c_n$. 
Maximize over $M$

For a state $\varrho$, the best precision is obtained with the operator given by the symmetric logarithmic derivative

$$M = 2i \sum_{k,l} \frac{\lambda_k - \lambda_l}{\lambda_k + \lambda_l} |k\rangle \langle l| \langle k| A|l\rangle,$$

where

$$\varrho = \sum_k \lambda_k |k\rangle \langle k|.$$
Metrological gain for given $c_1, c_2$ and for all $c_1, c_2$

- After several iterations of the two steps above, we obtain the maximal quantum Fisher information over a certain set of Hamiltonians.
- Based on that, we can calculate the quantity

$$g_{c_1, c_2}(\rho) = \max_{\mathcal{H}_1, \mathcal{H}_2} \frac{\mathcal{F}_Q(\rho, \mathcal{H}_1 \otimes \mathbb{I} + \mathbb{I} \otimes \mathcal{H}_2)}{\mathcal{F}_{Q}^{(\text{sep})}(c_1, c_2)},$$

where we assumed that $\mathcal{H}_n$ are constrained with

$$c_n \mathbb{I} \pm \mathcal{H}_n \geq 0.$$

- The separable limit for Hamiltonians of the form $\mathcal{H} = \mathcal{H}_1 \otimes \mathbb{I} + \mathbb{I} \otimes \mathcal{H}_2$ is

$$\mathcal{F}_{Q}^{(\text{sep})}(\mathcal{H}) = \sum_{n=1,2} [\sigma_{\max}(\mathcal{H}_n) - \sigma_{\min}(\mathcal{H}_n)]^2,$$

which leads to $\mathcal{F}_{Q}^{(\text{sep})}(c_1, c_2) = 4(c_1^2 + c_2^2)$. It is a constant.
Then, the gain can be expressed as

$$g(\varrho) = \max_{c_2} g_{c_1, c_2}(\varrho),$$

where the optimization is only over $c_2$, and, without the loss of generality, we set

$$c_1 = 1.$$

Why do we do it this way? Because direct maximization of

$$g = \frac{\mathcal{F}_Q[\varrho, \mathcal{H}]}{\mathcal{F}_{Q}^{(\text{sep})}(\mathcal{H})}$$

over the Hamiltonian is difficult, since both the numerator and the denominator must be maximized.
Convergence of the method

The precision cannot get worse with the iteration!
We remember that the $3 \times 3$ isotropic state is useful if

$$p < \frac{25 - \sqrt{177}}{32} \approx 0.3655.$$ 

Then, we have the following results for activation.

<table>
<thead>
<tr>
<th></th>
<th>Analytic example</th>
<th>Numerics</th>
</tr>
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<tbody>
<tr>
<td>Ancilla</td>
<td>0.3752</td>
<td>0.3941</td>
</tr>
<tr>
<td>Extra copy</td>
<td>0.4164</td>
<td>0.4170</td>
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</tbody>
</table>
Robustness of the metrological usefulness

\[ \varrho(p) = (1 - p)\varrho + p\varrho_{\text{noise}} \]

- Robustness of entanglement: the maximal \( p \) for which \( \varrho(p) \) is entangled for any separable \( \varrho_{\text{noise}} \).  
  [ Vidal and Tarrach, PRA 59, 141 (1999). ]

- Robustness of metrological usefulness: the maximal \( p \) for which \( \varrho(p) \) outperforms separable state for any separable \( \varrho_{\text{noise}} \).
Some PPT state can be activated by a pure product state

\[ \rho_{AB}^{(\text{PPT})} \otimes \rho_{A' B'}^{(\text{sep})}. \]

The optimal separable state is a pure product state \( \equiv 2 \) pure ancilla qudits.
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All bipartite pure entangled states are useful.

All entangled bipartite pure states are metrologically useful.

**Proof.**—For the two-qubit case, see P. Hyllus, O. Gühne, and A. Smerzi, Phys. Rev. A 82, 012337 (2010).

General case, pure state with a Schmidt decomposition

$$|\psi\rangle = \sum_{k=1}^{s} \sigma_k |k\rangle_A |k\rangle_B,$$

where $s$ is the Schmidt number, and the real positive $\sigma_k$ Schmidt coefficients are in a descending order.

We define

$$\mathcal{H}_A = \sum_{n=1,3,5,...,\tilde{s}-1} |+\rangle\langle+|A,n,n+1 - |-\rangle\langle-|A,n,n+1,$$

where $\tilde{s}$ is the largest even number for which $\tilde{s} \leq s$, and

$$|\pm\rangle_{A,n,n+1} = (|n\rangle_A \pm |n+1\rangle_A)/\sqrt{2}.$$
All bipartite pure entangled states are useful II

- We define $\mathcal{H}_B$ in a similar manner.

- We also define the collective Hamiltonian

$$\mathcal{H}_{AB} = \mathcal{H}_A \otimes 1 + 1 \otimes \mathcal{H}_B.$$  

Then, we have $\langle \mathcal{H}_{AB} \rangle_\psi = 0$.

- Direct calculation yields

$$\mathcal{F}_Q[|\psi\rangle, \mathcal{H}_{AB}] = 4(\Delta \mathcal{H}_{AB})^2_\psi = 8 \sum_{n=1,3,5,...,\tilde{s}-1} (\sigma_n + \sigma_{n+1})^2,$$

which is larger than the separable bound, $\mathcal{F}^{(sep)}_Q = 8$, whenever the Schmidt rank is larger than 1.
For even \( s \), this can be seen noting that

\[
\mathcal{F}_Q[^{\Psi}_\rangle, \mathcal{H}_{AB}] > 8 \sum_{n=1}^{s} \sigma_n^2
\]

holds, where we took into account that \( \sigma_n > 0 \) for \( n = 1, 2, 3, \ldots \), and \( \sum_{n=1}^{s} \sigma_n^2 = 1 \).

For odd \( s \), we need that

\[
\mathcal{F}_Q[^{\Psi}_\rangle, \mathcal{H}_{AB}] \geq 8 \left( \sum_{n=1}^{s-1} \sigma_n^2 + 2\sigma_1\sigma_2 \right) > 8 \sum_{n=1}^{s} \sigma_n^2
\]

holds, where we used that \( \sigma_1\sigma_2 > \sigma_s^2 \).
Infinite number of copies

- In the infinite copy limit, all bipartite pure entangled states are maximally useful.

- For the proof, see the [Supplemental Material of Tóth, Vértesi, Horodecki, Horodecki, PRL 2020. ]
Summary

- Some entangled quantum states that are not more useful for metrology than separable states can still be made more useful, if an ancilla or an additional copy is added.

- We have shown a general method to get the optimal local Hamiltonian for a quantum state.

- All bipartite pure entangled states are useful metrologically. If infinite number of copies are given, they are maximally useful.

See:

THANK YOU FOR YOUR ATTENTION!