Entanglement Detection in Continous Variable Systems

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Introduction

- In an experiment the density matrix is usually not known, only partial information is available on the quantum state. One can typically measure a few observables and still would like to detect some of the entangled states. Finding a criterion for entanglement with easily measurable observables is crucial for entanglement detection.
- There are only few such criteria in the literature. One of them is described in Ref. [1] for detecting entanglement in a two-mode system. One just has to measure the second moments of x and p for both systems. For example, if the inequality

$$(\Delta(x_A+x_B))^2+(\Delta(p_A-p_B))^2<2 \qquad (1)$$

is fulfilled, then the state is entangled [1].
- This criterion is equivalent to an entanglement witness if local unitary operations are allowed. A generalization of Ineq. (1) is a sufficent and necessary condition for entanglement of two-mode Gaussian states [1,2].

The detected state

- If one has N photons and sends them through a beam splitter or if one has N atoms in some internal state and applies a laser pulse, the state will be

$$|\Psi\rangle=\frac{1}{\sqrt{2^N N!}}(a^\dagger+b^\dagger)^N|0,0\rangle \qquad (2)$$

Here a and b are annihilation operators which are defined according to $x_A=(a+a^\dagger)/\sqrt{2}$. This state is not detected by the previous criterion as it will be demostrated later.
- We will present a criterion which: (i) requires measuring quantities which are easily accessible experimentally and; (ii) detects entangled states close to state (2).
- The criterion is quartic in operator expectation values and it cannot be reduced to an entanglement witness, even with the application of local unitary operations.

Entanglement criterion

- Our main result: For all separable states, i.e. states that can be written as

$$\rho=\sum_k p_k \rho_k^A\otimes\rho_k^B,$$

the following expression with the variances of the total particle number $N:=a^\dagger a+b^\dagger b$ and $(a-b)$ are bounded from below as

$$\left\{(\Delta_\rho N)^2+1\right\}\left\{(\Delta_\rho(a-b))^2+1\right\}\geq\frac{\langle N\rangle_\rho}{4}+\frac{1}{8}, \qquad (3)$$

where $(\Delta_\rho A)^2:=\langle A^\dagger A\rangle_\rho-|\langle A\rangle_\rho|^2$ (note that A need not be Hermitian).
- Physical motivation [3]: it is not possible to have a fixed particle number — corresponding to $(\Delta_\rho N)^2=0$ — and perfect interference — corresponding to $(\Delta_\rho(a-b))^2=0$ — at the same time, unless the system under consideration is in an entangled state. Only highly non-classical states can exhibit particle-like and wave-like features simultaneously.

Outline of proof

- The criterion is deduced from a simpler necessary condition for separability

$$w(\Delta_\rho N)^2+(1-w)(\Delta_\rho(a-b))^2\geq f_w(\langle N\rangle_\rho), \quad (4)$$

where $0<w<1$ and $f_w(N)$ is a monotonic function of N. All states violating this inequality are entangled.
- Ineq. (4) is based on a single-mode uncertainty relation

$$w(\Delta_\rho N_A)^2+(1-w)(\Delta_\rho a)^2\geq L_w(\langle N_A\rangle_\rho)\ ,$$

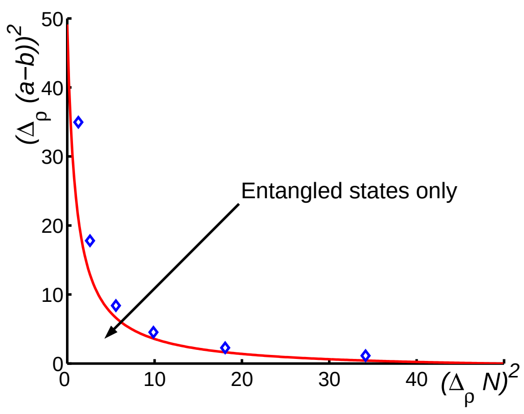
where $N_A=a^\dagger a$ and

$$L_w(N)=\sqrt{w(1-w)(N+\frac{1}{4})+\frac{w}{4}}-\frac{1}{2}.$$

The function $f_w(N)$ for Ineq. (4) can be obtained as $f_w(N)=L_w(N)+L(0)$.
- Our main result (3) is obtained by finding the region detected as entangled by (4) with any $w\in[0,1]$.

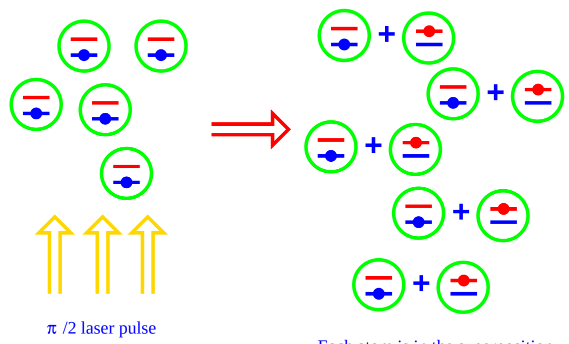
The $(\Delta N)^2$ – $(\Delta(a-b))^2$ plane

- Our method detects entangled states in the proximity of (2) on the $(\Delta_\rho N)^2-(\Delta_\rho(a-b))^2$ plane.
- Numerical verification of the inequality (3) for the two-mode separability problem. (red) Boundary of the region defined by Ineq. (3) for $N=200$. All states below this line are entangled. The $(0,0)$ point corresponds to the state (2). (blue) Points corresponding to separable states found numerically.



Realizations with BEC

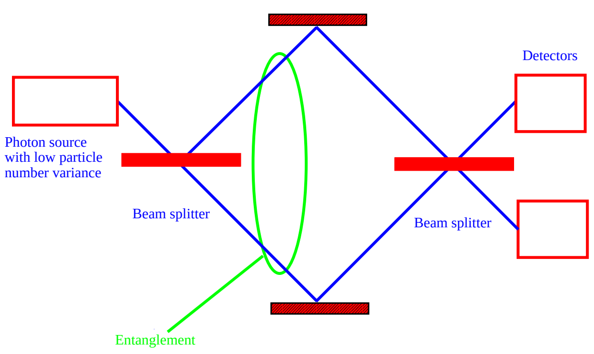
- The state (2) can be obtained in a Bose-Einstein condensate, by preparing the atoms in the same internal state and then applying a $\pi/2$ laser pulse.



- The entanglement between the modes a and b is physically much more meaningful, if the two modes are spatially separated. This could be be done for example by a state-dependent potential.

Realization with photons

- The state (2) can be prepared using a 50/50 beam splitter and a laser pulse corresponding to the $|\Psi\rangle\otimes|0\rangle$ state. After the second beam splitter ideally one gets back the $|\Psi\rangle\otimes|0\rangle$ state. The detectors measure the particle numbers in the two modes. In oder to detect entanglement, assuming perfect destructive interference at the second beam splitter, for the photon source $(\Delta N)^2\leq N/4-7/8$ is required. This requirement is satisfied, for example, by a number-squeezed coherent state.



Correlation matrix

- The entangled state (2) is not detected by the method based on the correlation matrix [1,2]. The correlation matrix γ contains the correlations of two pairs of conjugate single-party observables

$$\gamma_{kl} = Tr\{\rho(R_k-\langle R_k\rangle)(R_l-\langle R_l\rangle)\} + Tr\{\rho(R_l-\langle R_l\rangle)(R_k-\langle R_k\rangle)\}.$$

Here $\{R_k\}=\{x_A,p_A,x_B,p_B\}$, $x_A=(a+a^\dagger)/\sqrt{2}$, $p_A=(a-a^\dagger)/(\sqrt{2}i)$, and x_B and p_B are defined similarly for the b mode.
- The sufficient condition for inseparability is

$$T_a\gamma T_a-iJ\not\geq 0,$$

where $T_a\gamma T_a$ is the correlation matrix corresponding to the partially transposed density matrix and $J_{kl}=i[R_k,R_l]$.
- For the state (2) the $T_a\gamma T_a-iJ$ matrix is positive definite, thus the state is not detected as entangled.

Conclusions

- A simple inequality for the expectation values of observables was proposed for entanglement detection.
- Since only the measurement of easily accessible quantities (particle numbers and particle number variances) are needed, this approach may be feasible for detecting entanglement experimentally in Bose-Einstein condensates or with photons using linear optics.
- Other necessary conditions for separability could be constructed with the variances of two commuting operators. For example, entangled states close to the $|N,0\rangle+|0,N\rangle$ Schrödinger cat state could be detected by measuring the variances of N and $(a^\dagger b)^N+(ab^\dagger)^N$.

Related bibliography:

1. L.-M. Duan, G. Giedke, J.I. Cirac and P. Zoller, Phys. Rev. Lett. **84**, 2722 (2000).
2. R. Simon, Phys. Rev. Lett **84**, 2726(2000).
3. C. Simon, Phys. Rev. A **66**, 052323 (2002).