Permutationally invariant quantum tomography

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1 Motivation
   - Why quantum tomography is important?

2 Quantum experiments with multi-qubit systems
   - Physical systems
   - Local measurements

3 Full quantum state tomography
   - Basic ideas and scaling
   - Experiments
   - Approaches to solve the scalability problem

4 Permutationally invariant tomography
   - Main results
   - Example: XY PI tomography
   - Example: Experiment with a 4-qubit Dicke state

5 Extra slide 1: Number of settings
Why tomography is important?

- Many experiments aiming to create many-body entangled states.
- Quantum state tomography can be used to check how well the state has been prepared.
- However, the number of measurements scales exponentially with the number of qubits.
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Physical systems

State-of-the-art in experiments

- 14 qubits with trapped cold ions

- 10 qubits with photons
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Only local measurements are possible

**Definition**

A single **local measurement setting** is the basic unit of experimental effort.

A local setting means measuring operator $A^{(k)}$ at qubit $k$ for all qubits.

All two-qubit, three-qubit correlations, etc. can be obtained.

\[
\langle A^{(1)} A^{(2)} \rangle, \langle A^{(1)} A^{(3)} \rangle, \langle A^{(1)} A^{(2)} A^{(3)} \rangle, \ldots
\]
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Full quantum state tomography

- The density matrix can be reconstructed from $3^N$ measurement settings.

Example

For $N = 4$, the measurements are

1. $X$ $X$ $X$ $X$ $X$
2. $X$ $X$ $X$ $X$ $Y$
3. $X$ $X$ $X$ $Z$

...  

3$^4$. $Z$ $Z$ $Z$ $Z$ $Z$

- Note again that the number of measurements scales exponentially in $N$. 
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Experiments with ions and photons

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If the state is expected to be of a certain form (MPS), we can measure the parameters of the ansatz.
S.T. Flammia et al., arxiv:1002.3839; M. Cramer, M.B. Plenio, arxiv:1002.3780;
O. Landon-Cardinal et al., arxiv:1002.4632.

If the state is of low rank, we need fewer measurements.

We make tomography in a subspace of the density matrices (our approach).
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Permutationally invariant part of the density matrix:

$$\varrho_{PI} = \frac{1}{N!} \sum \Pi_k \varrho \Pi_k^\dagger,$$

where $\Pi_k$ are all the permutations of the qubits.


Main results

Features of our method:

1. Is for spatially separated qubits.
2. Needs the minimal number of measurement settings.
3. Uses the measurements that lead to the smallest uncertainty possible of the elements of $\rho_{PI}$.
4. Gives an uncertainty for the recovered expectation values and density matrix elements.
5. If $\rho_{PI}$ is entangled, so is $\rho$. Can be used for entanglement detection!
Measurements

- We measure the same observable $A_j$ on all qubits. (Necessary for optimality.)

- Each qubit observable is defined by the measurement directions $\vec{a}_j$ using $A_j = a_{j,x}X + a_{j,y}Y + a_{j,z}Z$.

Number of measurement settings:

$$D_N = \binom{N + 2}{N} = \frac{1}{2}(N^2 + 3N + 2).$$
What do we get from the measurements?

We obtain the expectation values for

$$\langle (A_j^\otimes (N-n) \otimes 1^\otimes n)_{\text{PI}} \rangle$$

for $j = 1, 2, \ldots, \mathcal{D}_N$ and $n = 0, 1, \ldots, N$. 
How do we obtain the Bloch vector elements?

A Bloch vector element can be obtained as

$$\langle (X^{\otimes k} \otimes Y^{\otimes l} \otimes Z^{\otimes m} \otimes 1^{\otimes n})_{\mathrm{Pl}} \rangle = \sum_{j=1}^{D_N} c_j^{(k,l,m)} \langle (A_j^{\otimes (N-n)} \otimes 1^{\otimes n})_{\mathrm{Pl}} \rangle.$$  

- Coefficients are not unique if $n > 0.$
Uncertainties

The uncertainty of the reconstructed Bloch vector element is

\[ \mathcal{E}^2[(X^k \otimes Y^l \otimes Z^m \otimes 1^n)_{\text{PI}}] = \sum_{j=1}^{D_N} |c_j^{(k,l,m)}|^2 \mathcal{E}^2[(A_j^{(N-n)} \otimes 1^n)_{\text{PI}}]. \]

For a fixed set of \( A_j \), we have a formula to find the \( c_j^{(k,l,m)} \)'s giving the minimal uncertainty.
Optimization for $A_j$

We have to find $D_N$ measurement directions $\vec{a}_j$ on the Bloch sphere minimizing the variance

$$(\mathcal{E}_{\text{total}})^2 = \sum_{k+l+m+n=N} \mathcal{E}^2 \left[ (X^{\otimes k} \otimes Y^{\otimes l} \otimes Z^{\otimes m} \otimes 1^{\otimes n})_{\text{PI}} \right] \times \left( \frac{N!}{k!l!m!n!} \right).$$
Summary of algorithm

Obtaining the "total uncertainty" for given measurements

\[ \rho_0, \text{ the state we expect} \]
\[ A_j, \text{ what we measure} \]
\[ \Rightarrow \text{BOX #1} \Rightarrow (\varepsilon_{\text{total}})^2 \]

Evaluating the experimental results

measurement results
\[ A_j \]
\[ \Rightarrow \text{BOX #2} \Rightarrow \left\{ \begin{align*}
\text{Bloch vector elements} \\
\text{variances}
\end{align*} \right\} \]
Estimation of the fidelity $F(\varrho, \varrho_{PI})$:

$$F(\varrho, \varrho_{PI}) \geq \langle P_s \rangle_{\varrho}^2 \equiv \langle P_s \rangle_{\varrho_{PI}}^2,$$

where $P_s$ is the projector to the $N$-qubit symmetric subspace.

- $F(\varrho, \varrho_{PI})$ can be estimated only from $\varrho_{PI}$!
- Proof: using the theory of angular momentum for qubits.
- Similar formalism appear concerning handling multi-copy qubit states:
  

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Let us assume that we want to know only the expectation values of operators of the form
\[ \langle A(\phi)^{\otimes N} \rangle \]
where
\[ A(\phi) = \cos(\phi)\sigma_x + \sin(\phi)\sigma_y. \]

The space of such operators has dimension \( N + 1 \). We have to choose \( \{\phi_j\}_{j=1}^{N+1} \) angles for the \( \{A_j\}_{j=1}^{N+1} \) operators we have to measure.
Simple example: XY PI tomography II

Let us assume that we measure

$$\langle A_j \otimes \xi \rangle$$

for \( j = 1, 2, \ldots, N + 1 \).

Reconstructed values and uncertainties

$$\langle A(\phi) \otimes \xi \rangle = \sum_{j=1}^{N+1} c_j^{(\phi)} \times \langle A_j \otimes \xi \rangle.$$  

Reconstructed coefficients  Measured data

$$\mathcal{E}^2 [A(\phi)] = \sum_{j=1}^{N+1} |c_j^{(\phi)}|^2 \mathcal{E}^2 (A_j \otimes \xi).$$

Let us assume that all of these measurements have a variance \( \Delta^2 \).
Simple example: XY PI tomography III

- Numerical example for $N = 6$.

Random directions $\phi_j$  Uncertainty of $A(\phi)^\otimes N$  Uniform directions
Numerical example for $N = 6$. This random choice is even worse ...
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4-qubit Dicke state, optimized settings (exp.)

(a) PI tomography with optimized settings vs. full tomography

(b) The measured correlations $\vec{a}_j$ measurement directions
Random settings (exp.)

(c) PI tomography with random settings vs. full tomography

(d) The measured correlations \( \vec{a}_j \) measurement directions
Density matrices (exp.)

a) full tomography
b) PI tomography optimized settings
c) PI tomography random settings

d) PI tomography optimized settings
f) PI tomography random settings
We determined the optimal $A_j$ for p.i. tomography for $N = 4, 6, ..., 14$. The maximal squared uncertainty of the Bloch vector elements is

$$\epsilon_{\text{max}}^2 = \max_{k,l,m,n} \mathcal{E}^2[(X^\otimes k \otimes Y^\otimes l \otimes Z^\otimes m \otimes I^\otimes n)_{\text{PI}}]$$

(Total count is the same as in the experiment: 2050.)
Expectation values directly from measured data

- Operator expectation values can be recovered directly from the measurement data as

\[ \langle Op \rangle = \sum_{j=1}^{D_N} \sum_{n=1}^{N} c_{j,n}^{Op} \langle (A_j^{\otimes (N-n)} \otimes \mathbb{1}^\otimes n)_{PI} \rangle, \]

where the \(c_{j,n}^{Op}\) are constants.

- \(Op = |D^{(N/2)}_N\rangle\langle D^{(N/2)}_N|\), blue: \(\rho_0 \propto \mathbb{1}\), red: upper bound for any \(\rho_0\).
Comparison with other methods for efficient tomography

- If a state is detected as entangled, it is surely entangled. No assumption is used concerning the form of the quantum state.

- Expectation values of all permutationally invariant operators are the same for $\rho$ and $\rho_{PI}$.

- Typically, it can be used in experiments in which permutationally invariant states are created.
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Summary

- We discussed permutationally invariant tomography for large multi-qubits systems.
- It paves the way for quantum experiments with more than 6 – 8 qubits.

See:

THANK YOU FOR YOUR ATTENTION!
How many settings we need?

- Expectation values of \((X^{\otimes k} \otimes Y^{\otimes l} \otimes Z^{\otimes m} \otimes 1^{\otimes n})_{PI}\) are needed.

- For a given \(n\), the dimension of this subspace is \(\mathcal{D}_{(N-n)}\) (simple counting).

- Operators with different \(n\) are orthogonal to each other.

- Every measurement setting gives a single real degree of freedom for each subspace.

- Hence the number of settings cannot be smaller than the largest dimension, which is \(\mathcal{D}_N\).