Detecting $k$-particle entanglement with spin squeezing inequalities
(a derivation from arxiv:1104.3147,
talk by G. Vitagliano)

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5 States maximally violating it

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Why $k$-particle entanglement is important?

- Many experiments are aiming to create many-body entangled states.
- It is not sufficient to say “entangled”. We have to say something like “genuine multipartite entangled”.
- In experiments with a million atoms, we can only measure collective quantities.

See also
[ L.-M. Duan, *Entanglement detection in the vicinity of arbitrary Dicke states*, arXiv:1107.5162 ],
## Genuine multipartite entanglement

### Definition

A state is (fully) separable if it can be written as

$$\sum_k p_k \rho_1^{(k)} \otimes \rho_2^{(k)} \otimes \ldots \otimes \rho_N^{(k)}.$$  

### Definition

A pure multi-qubit quantum state is called biseparable if it can be written as the tensor product of two multi-qubit states

$$|\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle.$$  

Here $|\psi\rangle$ is an $N$-qubit state. A mixed state is called biseparable, if it can be obtained by mixing pure biseparable states.

### Definition

If a state is not biseparable then it is called genuine multi-partite entangled.
**Definition**

A pure state is \( k \)-producible if it can be written as

\[
|\Phi\rangle = |\Phi_1\rangle \otimes |\Phi_2\rangle \otimes |\Phi_3\rangle \otimes |\Phi_4\rangle \ldots
\]

where \( |\Phi_i\rangle \) are states of at most \( k \) qubits. A mixed state is \( k \)-producible, if it is a mixture of \( k \)-producible pure states.


- In many-particle systems where only collective quantities can be detected, this is the only meaningful characterization of entanglement.

- That is, genuine multipartite entanglement is very difficult to detect in such systems.
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Physical systems

State-of-the-art in experiments

- 100,000 atoms realizing an array of 1D Ising spin chains (Nature, 2003)
- Spin squeezing with $10^6 - 10^{12}$ atoms (Nature, 2001)

Main challenge

- The particles cannot be addressed individually.
- Only collective quantities can be measured.
- New type of entangled states and entanglement criteria are needed.
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For spin-$\frac{1}{2}$ particles, we can measure the collective angular momentum operators:

$$J_l := \frac{1}{2} \sum_{k=1}^{N} \sigma_l^{(k)},$$

where $l = x, y, z$ and $\sigma_l^{(k)}$ a Pauli spin matrices.

We can also measure the

$$(\Delta J_l)^2 := \langle J_l^2 \rangle - \langle J_l \rangle^2$$

variances.
For spin-$j$ particles for $j > 1/2$, we can measure the collective angular momentum operators:

$$G_l := \sum_{k=1}^{N} g_l^{(k)},$$

where $l = 1, 2, \ldots, d^2 - 1$ and $g_l^{(k)}$ are the SU(d) generators.

We can also measure the

$$(\Delta G_l)^2 := \langle G_l^2 \rangle - \langle G_l \rangle^2$$

variances.
Only collective measurements are possible

A condition for separability is

\[ \sum_k (\Delta G_k)^2 \geq 2N(d - 1). \]

Maximally violating states

For \( N = d \), the multipartite singlet state maximally violates the condition with \( \sum_k (\Delta G_k)^2 = 0 \).

For \( N < d \), there is no quantum states for which \( \sum_k (\Delta G_k)^2 = 0 \).

This can be seen as follows. It is not possible to create a completely antisymmetric state of \( d \)-state particles with less than \( d \) particles.
Maximally violating states II

A more detailed proof: For the sum of the squares of $G_k$ we obtain

$$\sum_k (G_k)^2 = \sum_k \sum_n (g_k^{(n)})^2 + \sum_k \sum_{n \neq m} g_k^{(m)} g_k^{(n)}$$

$$= 2N \frac{d^2 - 1}{d} 1 + \sum_{n \neq m} 2 \left( F_{mn} - \frac{1}{d} \right).$$

Based on this and using $\langle F_{mn} \rangle \geq -1$, we can write

$$\sum_k \langle(G_k)^2\rangle \geq \frac{2N}{d} (d + 1)(d - N).$$

The bound on the right-hand side cannot be zero if $N < d$.

For $N = d$, the sum $\sum_k \langle(G_k)^2\rangle$ is zero for the totally antisymmetric state for which $\langle F_{mn} \rangle = -1$ for all $m, n$. 
Maximally violating states III

It can also be proved that

\[ \sum_k \langle G_k^2 \rangle = 0 \iff \sum_k (\Delta G_k)^2 = 0. \]

Two-producibility

We look for the minimum of

$$\sum_k (\Delta G_k)^2 = \sum_k \langle G_k^2 \rangle - \sum_k \langle G_k \rangle^2.$$ 

Let us see a two-particle system. We will compute the minimum/maximum for both terms.
First term

- First, let us see \[ \sum_k \langle G_k^2 \rangle. \]

- We have to consider symmetric and antisymmetric states. The inequality is saturated for symmetric states.
What do we have for antisymmetric states?

\[ \sum_k \left\langle G_k^2 \right\rangle = \sum_k \left\langle (g_k^{(1)})^2 \right\rangle + \sum_k \left\langle (g_k^{(2)})^2 \right\rangle + 2 \sum_k \left\langle (g_k^{(1)})(g_k^{(2)}) \right\rangle. \]

Here

\[ \left\langle \sum_k (g_k^{(1)})^2 \right\rangle = \left\langle \sum_k (g_k^{(2)})^2 \right\rangle = 2(d + 1)(1 - 1/d). \]

And,

\[ \left\langle \sum_k (g_k^{(1)})(g_k^{(2)}) \right\rangle = -2(1 + 1/d). \]

(This is because with the flip operator \( F \) we can be write as \( \sum_k g_k^{(1)} g_k^{(2)} = 2F - \frac{2}{d} \).

Then, we obtain

\[ \sum_k \left\langle G_k^2 \right\rangle = 4(d + 1)(1 - 2/d). \]
Then, one has to deal with $\sum_k \langle G_k \rangle^2$. For that, we get
\[
\sum_k \langle G_k \rangle^2 = \sum_k \langle g_k^{(1)} + g_k^{(2)} \rangle^2 = \sum_k \langle g_k^{(1)} \rangle^2 + \sum_k \langle g_k^{(2)} \rangle^2 + 2M,
\]
where
\[
M = \sum_k \langle g_k^{(1)} \rangle \langle g_k^{(2)} \rangle.
\]

Knowing that
\[
\sum_k \langle g_k^{(n)} \rangle^2 \leq 2(1 - 1/d).
\]
and using the Cauchy-Schwarz inequality one gets
\[
\sum_k \langle G_k \rangle^2 \leq 8(1 - 1/d).
\]

Now, we have to use again that for a single qudit
\[
\sum_k \langle g_k^{(n)} \rangle^2 = 2\text{Tr}(\rho^2) - 2/d.
\]
Lemma.

For bipartite antisymmetric states we have

\[ \text{Tr}(\rho_{\text{red}}^2) \leq \frac{1}{2}. \]

Proof. All pure two-qudit antisymmetric states can be written in some basis as

\[ \alpha_{12}|\psi_{12}^-\rangle + \alpha_{34}|\psi_{34}^-\rangle + \alpha_{56}|\psi_{56}^-\rangle + \ldots, \]

where \( \alpha_{nm} \) are constants and

\[ \psi_{mn}^- = (|mn\rangle - |nm\rangle)/\sqrt{2}. \]

Then for the collective operators for antisymmetric states we have

\[ \sum_k \langle G_k \rangle^2 \leq 4(1 - 2/d) = 4 - 8/d. \]
Symmetric and antisymmetric states

- Hence, for antisymmetric states, one gets

\[ \sum_k (\Delta G_k)^2 \geq 4d(1 - 2/d) = 4(d - 2) = 4d - 8. \]

- For symmetric states, we get

\[ \sum_k (\Delta G_k)^2 \geq 4(1 - \frac{1}{d})(2 + d) - 8(1 - 1/d) = 8d(1 - \frac{1}{d}) = 8d - 8. \]

This bound is always larger than the one for antisymmetric states.
Lemma. We know that

\[ \sum_k (\Delta G_k)^2 = \sum_k (\Delta G_k)^2_{\rho'} \]

where

\[ \rho' = P_a \rho P_a + P_s \rho P_s. \]

It is the same as

\[ \rho' = \frac{1}{2}(\rho + F \rho F), \]

where \( F \) is the flip operator. Hence, the coherences between the symmetric and asymmetric parts need not be considered.

Proof. The variance of a collective operator is permutationally invariant.
The criterion

A condition for two-producibility for $N$ qudits of dimension $d$ is

$$\sum_k (\Delta G_k)^2 \geq 2N(d - 2).$$

A condition for separability is

$$\sum_k (\Delta G_k)^2 \geq 2N(d - 1).$$
Summary

- We showed that a certain generalized spin-squeezing inequality can be used to detect three-particle entanglement.
- The inequality detects states close to many-body singlet states.

See:

THANK YOU FOR YOUR ATTENTION!