Spin-squeezing inequalities for entanglement detection in cold gases

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Outline

1 Motivation
   - Why spin squeezing inequalities are important?

2 Physical systems
   - Cold gases

3 Multipartite entanglement
   - Definition of entanglement

4 Spin squeezing entanglement criteria for $j = 1/2$
   - Collective measurements
   - The original criterion
   - Generalized criteria for $j = \frac{1}{2}$

5 Spin squeezing inequality for an ensemble of spin-$j$ atoms
   - Conditions with the angular momentum coordinates for $j > \frac{1}{2}$
   - The usual spin squeezing inequality for $j > \frac{1}{2}$
   - Conditions with the SU(d) generators
   - Detection of singlets
Many experiments are aiming to create entangled states with many atoms.

Only collective quantities can be measured.

Most experiments use atoms with $j > \frac{1}{2}$. 
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Physical systems

State-of-the-art in experiments

- 100,000 atoms realizing an array of 1D Ising spin chains (Nature, 2003)
- Spin squeezing with $10^6 - 10^{12}$ atoms (Nature, 2001)

Main challenge

- The particles cannot be addressed individually.
- Only collective quantities can be measured.
- New type of entangled states and entanglement criteria are needed.
For example: Spin squeezing in a cold atomic ensemble

Picture from M.W. Mitchell, ICFO, Barcelona.
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Entanglement

Definition

A multiparticle state is (fully) separable if it can be written as

$$\sum_k p_k \rho_1^{(k)} \otimes \rho_2^{(k)} \otimes \ldots \otimes \rho_N^{(k)}.$$ 

If a state is not fully separable, then it is called entangled.
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For spin-\( \frac{1}{2} \) particles, we can measure the collective angular momentum operators:

\[
J_l := \frac{1}{2} \sum_{k=1}^{N} \sigma^{(k)}_l,
\]

where \( l = x, y, z \) and \( \sigma^{(k)}_l \) a Pauli spin matrices.

We can also measure the variances

\[
(\Delta J_l)^2 := \langle J_l^2 \rangle - \langle J_l \rangle^2.
\]
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The standard spin-squeezing criterion

- The spin squeezing criteria for entanglement detection is

\[
\frac{(\Delta J_x)^2}{\langle J_y \rangle^2 + \langle J_z \rangle^2} \geq \frac{1}{N}.
\]

- If it is violated then the state is entangled.

[A. Sørensen, L.M. Duan, J.I. Cirac, P. Zoller, Nature 409, 63 (2001).]

- States violating it are like this:
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Let us assume that for a system we know only

\[ \vec{J} := (\langle J_x \rangle, \langle J_y \rangle, \langle J_z \rangle), \]
\[ \vec{K} := (\langle J_x^2 \rangle, \langle J_y^2 \rangle, \langle J_z^2 \rangle). \]

Then any state violating the following inequalities is entangled.

\[ \langle J_x^2 \rangle + \langle J_y^2 \rangle + \langle J_z^2 \rangle \leq \frac{N(N+2)}{4}, \]
\[ (\Delta J_x)^2 + (\Delta J_y)^2 + (\Delta J_z)^2 \geq \frac{N}{2}, \]
\[ \langle J_k^2 \rangle + \langle J_l^2 \rangle \leq (N - 1)(\Delta J_m)^2 + \frac{N}{2}, \]
\[ (N - 1)\left[(\Delta J_k)^2 + (\Delta J_l)^2\right] \geq \langle J_m^2 \rangle + \frac{N(N-2)}{4}, \]

where \( k, l, m \) take all the possible permutations of \( x, y, z \).

Generalized spin squeezing criteria for $j = \frac{1}{2}$

- The previous inequalities, for fixed $\langle J_{x/y/z} \rangle$, describe a polytope in the $\langle J_{x/y/z}^2 \rangle$ space.

- For $\langle \vec{J} \rangle = 0$ and $N = 6$ the polytope is the following:
Completeness

- Random separable states:

- The **completeness** can be proved for large $N$. 
The polytope for $N = 10$ and $J = (0, 0, 0), \quad J = (0, 0, 2.5), \quad J = (0, 0, 4.5)$. 
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For the $j = \frac{1}{2}$ case, the SSIs were developed based on the first and second moments and variances of the such collective operators.

For the $j > \frac{1}{2}$ case, we define the modified second moment

$$\langle \tilde{J}_k^2 \rangle := \langle J_k^2 \rangle - \langle \sum_n (j_k^{(n)})^2 \rangle = \sum_{m \neq n} \langle j_k^{(n)} j_k^{(m)} \rangle$$

and the modified variance

$$\langle \tilde{\Delta}J_k \rangle^2 := \langle \Delta J_k \rangle^2 - \langle \sum_n (j_k^{(n)})^2 \rangle.$$  

These are essential to get tight equations for $j > \frac{1}{2}$. 

“Modified” quantities for $j > \frac{1}{2}$
The inequalities for $j > \frac{1}{2}$ with the angular momentum coordinates

For fully separable states of spin-$j$ particles, all the following inequalities are fulfilled

$$
\langle J_x^2 \rangle + \langle J_y^2 \rangle + \langle J_z^2 \rangle \leq Nj(Nj + 1),
$$

$$
(\Delta J_x)^2 + (\Delta J_y)^2 + (\Delta J_z)^2 \geq Nj,
$$

$$
\langle \tilde{J}_k^2 \rangle + \langle \tilde{J}_l^2 \rangle - N(N - 1)j^2 \leq (N - 1)(\tilde{\Delta}J_m)^2,
$$

$$
(N - 1) \left[ (\tilde{\Delta}J_k)^2 + (\tilde{\Delta}J_l)^2 \right] \geq \langle \tilde{J}_m^2 \rangle - N(N - 1)j^2,
$$

where $k, l, m$ take all possible permutations of $x, y, z$.

Violation of any of the inequalities implies entanglement.
In the large $N$ limit, the inequalities with the angular momentum are complete.

It is not possible to find new entanglement conditions based on $\langle J_k \rangle$ and $\langle \tilde{J}_k^2 \rangle$ that detect more states.
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The usual spin squeezing inequality for $j > \frac{1}{2}$

The standard spin-squeezing inequality becomes

$$\frac{(\Delta J_x)^2}{\langle J_y \rangle^2 + \langle J_z \rangle^2} + \frac{\sum_n(J^2 - \langle (J^{(n)}_x)^2 \rangle)}{\langle J_y \rangle^2 + \langle J_z \rangle^2} \geq \frac{1}{N}.$$ 

Violated only if there is entanglement between the spin-$j$ particles.

The second term on the LHS is nonnegative.
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The inequalities for \( j > \frac{1}{2} \) with the \( G_k \)'s

Collective operators:

\[
G_l := \sum_{k=1}^{N} g^{(k)}_l,
\]

where \( l = 1, 2, ..., d^2 - 1 \) and \( g^{(k)}_l \) are the SU(d) generators.

We can also measure the

\[
(\Delta G_l)^2 := \langle G^2_l \rangle - \langle G_l \rangle^2
\]

variances.
The inequalities for $j > \frac{1}{2}$ with the $G_k$'s

- The SSIs for $G_k$ have the general form

$$
(N - 1) \sum_{k \in I} (\tilde{\Delta} G_k)^2 - \sum_{k \notin I} \langle (\tilde{G}_k)^2 \rangle \geq -2N(N - 1) \frac{(d - 1)}{d}.
$$

- For instance, for the $d = 3$ case, the SU(d) generators can be the eight Gell-Mann matrices.

- $I$ is a subset of indices between 1 and $M$. We have $2^M$ equations!
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An example: The criterion for SU(d) singlets

A condition for two-producibility (i.e., higher form of entanglement) for $N$ qudits of dimension $d$ is

$$\sum_k (\Delta G_k)^2 \geq 2N(d - 2).$$

A condition for separability is

$$\sum_k (\Delta G_k)^2 \geq 2N(d - 1).$$

[ G. Vitagliano, P. Hyllus, I.L. Egusquiza, and G. Tóth, Spin squeezing inequalities for arbitrary spin, PRL 2011. ]
### Group

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<tr>
<th>Name</th>
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### Topics
- Multipartite entanglement and its detection
- Metrology, cold gases
- Collaborating on experiments:
  - Weinfurter group, Munich, (photons)
  - Mitchell group, Barcelona, (cold gases)

### Funding:
- European Research Council starting grant 2011-2016, 1.3 million euros
- CHIST-ERA QUASAR collaborative EU project
- Grants of the Spanish Government and the Basque Government
G. Vitagliano, Spin squeezing and entanglement for arbitrary spin (more details)

I. Urizar-Lanz, Differential magnetometry using singlets

I. Apellaniz, Accuracy Bounds for Gradient Metrology in Atomic Ensembles
Links to a QIPC Talk

H. Weinfurter, Tuesday 11:30-12:15, Analyzing multi-qubit quantum states – Permutationally Invariant Tomography.

Permutationally invariant tomography

- Full state tomography is not feasible even for modest size systems.
- PI tomography is a scaleable alternative (PRL 2010).
- We developed a scaleable method for fitting a physical density matrix on the measured data (before: bottleneck of state reconstruction)

Ongoing experiment at the Max Planck Institute for Quantum Optics, München.

G. Tóth et al., Phys. Rev. Lett. 105, 250403 (2010);
Summary

- Full set of generalized spin squeezing inequalities with $J_l$ with $l = x, y, z$ for $j > \frac{1}{2}$.
- Large set of inequalities with the other collective operators.
- These might make possible new experiments and make existing experiments simpler.


See www.gtoth.eu for the slides

THANK YOU FOR YOUR ATTENTION!