Quantum entanglement and its detection with few measurements

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Outline

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Introduction

- Rapid development of quantum engineering and quantum control:
  - Few particles (< 10), creation of interesting quantum states in various physical systems, such as trapped ions, photonic systems, or molecules controlled by nuclear magnetic resonance (NMR).
  - Large scale (e.g., 10^5 particles) systems, for example, optical lattices of cold two-state atoms and cold atomic clouds.
- These experiments are possible due to novel technologies developed in the last ten years.
- Quantum information science and the theory of entanglement helps identifying the quantum states that are highly nonclassical.

Outline

EPR paradox

- Quantum mechanics is very different from classical physics.
- There are qualitatively new and counterintuitive *two-body* and *many-body* phenomena.
- One of the earliest study was in

MAY 15, 1935 PHYSICAL REVIEW VOLUME 47

Can Quantum-Mechanical Description of Physical Reality Be Considered Complete?

A. Einstein, B. Podolsky and N. Rosen, *Institute for Advanced Study, Princeton, New Jersey*

(Received March 25, 1935)

In a complete theory there is an element corresponding to each element of reality. A sufficient condition for the reality of a physical quantity is the possibility of predicting it with certainty, without disturbing the system. In quantum mechanics in the case of two physical quantities described by non-commuting operators, the knowledge of one precludes the knowledge of the other. Then either (1) the description of reality given by the wave function in quantum mechanics is not complete or (2) these two quantities cannot have simultaneous reality. Consideration of the problem of making predictions concerning a system on the basis of measurements made on another system that had previously interacted with it leads to the result that if (1) is false then (2) is also false. One is thus led to conclude that the description of reality as given by a wave function is not complete.

EPR paradox II

- The paper considered two particles in a singlet state

$$|\Psi_{\text{singlet}}\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle).$$

- Let us call the two parties A and B (Alice and Bob).

- Some simple measurement scenarios:

Alice	Bob
$z = +1$	$z = -1$
$z = -1$	$z = +1$
$x = +1$	$z = \pm 1$

EPR paradox III

- How does Bob's particle know, what Alice measured? Is not it action at a distance?)
- The outcome is random in some cases. We should be able to determine the outcome of the measurement.
- Maybe, we just do not have enough information. There can be sofar unknown elements of reality that determine the measurement outcome.

Outline

Bell inequalities

- 30 years later appeared a paper that formulated the EPR paradox in a qualitative way.

Physics Vol. 1, No. 3, pp. 195–200, 1964 Physics Publishing Co. Printed in the United States

ON THE EINSTEIN PODOLSKY ROSEN PARADOX*

J. S. BELL†
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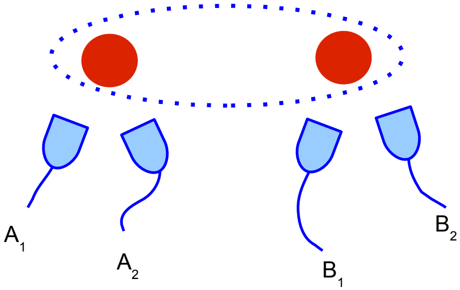
(*Received* 4 *November* 1964)

I. Introduction

THE paradox of Einstein, Podolsky and Rosen [1] was advanced as an argument that quantum mechanics could not be a complete theory but should be supplemented by additional variables. These additional variables were to restore to the theory causality and locality [2]. In this note that idea will be formulated

Local hidden variable (LHV) models

- Do the measured quantities correspond to an element of reality before the measurement? Let us assume that they do.
- Let us see the bipartite case. Assume that we measure A_1 and A_1 at party A, and measure B_2 and B_2 at party B. Both A_k and B_k have ± 1 measurement results.



Local hidden variable (LHV) models II

- A_k and B_k are quantum mechanically incompatible.
- Let us assume that all the four measurement outcomes exist before the measurement.
- The idea is that at each measurement k, there are $a_{1,k}, a_{2,k}, b_{1,k}, b_{2,k}$ available.
- We expect a measurement record like the following:

k	$a_{1,k}$	$a_{2,k}$	$b_{1,k}$	$b_{2,k}$
1	$+1$	-1	$+1$	$+1$
2	-1	$+1$	$+1$	-1
3	$+1$	$+1$	-1	$+1$
4	-1	-1	$+1$	-1
5	$+1$	$+1$	$+1$	-1
6	-1	-1	-1	$+1$
...	...	...	...	...

Red color indicates the measured values.

Local hidden variable (LHV) models III

- The correlations can be obtained as

$$\langle A_m B_n \rangle = \frac{1}{M} \sum_{k=1}^{M} a_{m,k} b_{n,k}.$$

- Here, k is the hidden variable.
- Usual formula, with λ as a hidden variable

$$f(a_m, b_n) = \int f_{m,\lambda}(a_m) g_{n,\lambda}(b_n) d\lambda$$

  Here f's and $g's$ are probability density functions.
- In words: all two-variable probability distributions can be given as a sum of product distributions.

Bipartite nonlocality

- Let us consider the following expression:

$$A_1 B_1 + A_2 B_1 + A_1 B_2 - A_2 B_2.$$

- Let us now substitute $+1$ or -1 to A_k and B_k. There are 16 combinations. We obtain

$$A_1 B_1 + A_2 B_1 + A_1 B_2 - A_2 B_2 \leq 2$$

- But, if we identify A with σ_x and B with σ_y, then there is a quantum state for which

$$\left\langle \sigma_x \otimes \sigma_x + \sigma_y \otimes \sigma_x + \sigma_x \otimes \sigma_y - \sigma_y \otimes \sigma_y \right\rangle = 2\sqrt{2}.$$

  This state is, apart from local transformations, the singlet $|01\rangle - |10\rangle$.

Bipartite nonlocality II

- The real measurement record is the following:

k	$a_{1,k}$	$a_{2,k}$	$b_{1,k}$	$b_{2,k}$
1	$+1$	...	$+1$	...
2	-1	...	...	-1
3	...	$+1$	-1	...
4	-1	...	$+1$	...
5	...	$+1$	...	-1
6	-1	...	-1	...
...	...	...	...	...

Red color indicates the measured values.

- The correlations can be obtained as

$$\langle A_m B_n \rangle = \frac{1}{|\mathcal{M}_{m,n}|} \sum_{k \in \mathcal{M}_{m,n}} a_{m,k} b_{n,k},$$

where $\mathcal{M}_{m,n}$ contains the indices corresponding to measuring A_m and B_n. This is the reason that correlations do not fit an LHV model.

Bipartite nonlocality III

Definition

Bell inequalities are inequalities with correlation terms that are constructed to exclude LHV models. They have the form

$$\langle \mathcal{B} \rangle \leq \text{const.},$$

where $\mathcal{B}$ is the Bell operator.

- One of the first one was the CHSH inequality,

$$A_1 B_1 + A_2 B_1 + A_1 B_2 - A_2 B_2 \leq 2.$$

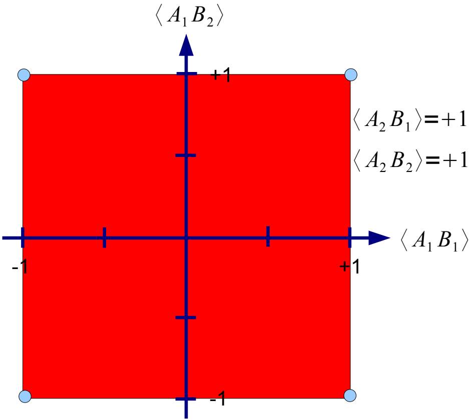
Definition

The visibility of a Bell inequality is defined as

$$\mathcal{V}(\mathcal{B}) = \frac{\max_{\Psi} \langle \mathcal{B} \rangle_{\Psi}}{\max_{\mathrm{LHV}} \mathcal{B}}.$$

Convex sets: Correlations compatible with LHV models

- The points corresponding to correlations fulfilling Bell inequalities are within a polytope. Extreme points have correlations ± 1



Outline

Multipartite nonlocality (GHZ, 1989)

- There are also multipartite Bell inequalities. For the multipartite case, quantum mechanics violates locality even on an all versus nothing basis.

Definition

Greenberger-Horne-Zelinger(GHZ) state

$$|GHZ\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle).$$

- We measure Pauli spin matrices X and Y at all qubits.
- If we flip all qubits ($|0\rangle \leftrightarrow |1\rangle$), we get back the GHZ state

$$\langle X_1 X_2 X_3\rangle = \langle GHZ|X_1 X_2 X_3|GHZ\rangle = +1.$$

Multipartite nonlocality (GHZ, 1989) II

- If we flip one qubit ($|0\rangle \leftrightarrow |1\rangle$) and apply flip+phase shift for the other two, we get back the GHZ state

$$\langle X_1 Y_2 Y_3 \rangle = \langle GHZ | X_1 Y_2 Y_3 | GHZ \rangle = -1.$$

- We also have $\langle Y_1 X_2 Y_3 \rangle = \langle Y_1 Y_2 X_3 \rangle = -1$.
- Based on common sense we would expect

$$X_1 X_2 X_3 = (Y_1 Y_2 X_3)(Y_1 X_2 Y_3)(X_1 Y_2 Y_3) = -1 (wrong)$$

However, this is wrong. $\langle X_1 X_2 X_3 \rangle = +1$ for the GHZ state.

- Not only statistical contradiction. All experiments contradict the assumption of an LHV model.

[D. M. Greenberger, M. Horne, and A. Zeilinger, 1989.]

Mermin's inequality (N.D. Mermin, PRL 1990)

For N qubits, the Mermin inequality is given by

$$\sum_{\pi}\langle X_1 X_2 X_3 X_4 X_5 \cdots X_N\rangle - \sum_{\pi}\langle Y_1 Y_2 X_3 X_4 X_5 \cdots X_N\rangle + \sum_{\pi}\langle Y_1 Y_2 Y_3 Y_4 X_5 \cdots X_N\rangle - \ldots + \ldots \leq L_{\mathrm{Mermin}},$$

where $\sum_{\pi}$ represents the sum of all possible permutations of the qubits that give distinct terms.

- L_{Mermin} is the maximum for local states. It is defined as

$$L_{\mathrm{Mermin}} = \begin{cases} 2^{N/2} & \text{for even } N, \\ 2^{(N-1)/2} & \text{for odd } N. \end{cases}$$

- The quantum maximum is 2^{N-1}.

Mermin's inequality II

- The visibility is increases exponentially with the number of qubits:

$$\mathcal{V}_{\mathrm{Mermin}} = \begin{cases} 2^{N/2-1} & \text{for even } N, \\ 2^{N/2-1/2} & \text{for odd } N. \end{cases}$$

Ardehali's inequality (M. Ardehali, PRA 1992)

Definition

The Ardehali inequality

$$\langle (A_1^{(+)} - A_1^{(-)})\Big(-\sum_\pi X_2X_3X_4X_5\cdot\cdot X_N + \sum_\pi Y_2Y_3X_4X_5\cdot\cdot X_N - \sum_\pi Y_2Y_3Y_4Y_5X_6\cdot\cdot X_N + \ldots - \ldots\Big)\rangle$$
$$+\langle (A_1^{(+)} + A_1^{(-)})\Big(\sum_\pi Y_2X_3X_4X_5\cdot\cdot X_N - \sum_\pi Y_2Y_3Y_4X_5\cdot\cdot X_N + \sum_\pi X_2Y_3Y_4Y_5Y_6X_7\cdot\cdot X_N - \ldots + \ldots\Big)\rangle \leq L_{\text{Ardehali}},$$

where $A_1^{(\pm)}$ are operators corresponding to measuring the first spin along directions corresponding to the quantum operators $A_1^{(\pm)} = (\mp X_1 - Y_1)/\sqrt{2}$.

Ardehali's inequality II

- The Ardehali's inequality is again maximally violated by the GHZ state.
- The constant is

$$L_{\text{Ardehali}} = \begin{cases} 2^{N/2} & \text{for even } N, \\ 2^{(N+1)/2} & \text{for odd } N. \end{cases}$$

- The quantum maximum is $2^{N-1} \times \sqrt{2} = 2^{N-1/2}$.
- The visibility increases exponentially with the number of qubits:

$$\mathcal{V}_{\text{Ardehali}} = \begin{cases} 2^{N/2-1/2} & \text{for even } N, \\ 2^{N/2-1} & \text{for odd } N. \end{cases}$$

- Remember:

$$\mathcal{V}_{\text{Mermin}} = \begin{cases} 2^{N/2-1} & \text{for even } N, \\ 2^{N/2-1/2} & \text{for odd } N. \end{cases}$$

Bell inequalities with full correlation terms

Definition

A full correlation term contains a variable for each spin.

- $X_1 Y_2 X_3 Y_4$ is a full correlation term
- $X_1 Y_2 \mathbb{1}_3 X_4$ is not.

- Among inequalities with full correlations terms, for any N, the Mermin-Ardehali construction has the largest violation possible.
- The full set of such Bell inequalities can be written down concisely in the form of a single nonlinear inequality. [R.F.Werner, M.Wolf, PRA 2001; M. Zukowski, C. Brukner, PRL 2002.]
- There are multi-qubit pure entangled states that do not violate any of these Bell inequalities. [M. Zukowski *et al.*, PRL 2002.]

Not full correlation terms

- It has been shown that such inequalities can detect any pure entangled multi-qubit state.

  [S. Popescu, D. Rohrlich, PLA 1992.]

- Inequalities of this type can be constructed such that they are maximally violated by cluster states and graph states. In particular, for the four-qubit cluster state this inequality looks like

$$\langle X_1 \mathbb{1}_2 X_3 Z_4 \rangle + \langle Z_1 Y_2 Y_3 Z_4 \rangle + \langle X_1 \mathbb{1}_2 Y_3 Y_4 \rangle - \langle Z_1 Y_2 X_3 Y_4 \rangle \leq 2.$$

  On all of the qubits two operators are measured except for the second qubit for which only Y_2 is measured.

- For a large class of graph states, e.g., for linear cluster states, it is possible to construct two-setting Bell inequalities that have a visibility increasing exponentially with N. [O. Gühne *et al.*, PRL 2005; G. Tóth *et al.*, PRA 2006.]

Experiment

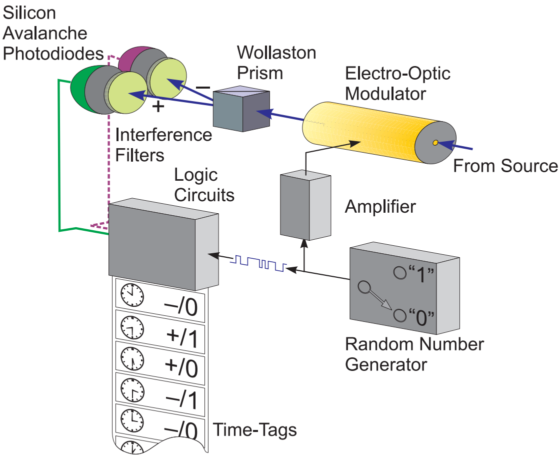


Figure: One of the two observer stations. All alignments and adjustments were pure local operations that did not rely on a common source or on communication between the observers.

[Figure from G. Weihs *et al.*, Phys. Rev. Lett. **81** 5039 (1998).]

Experiment II

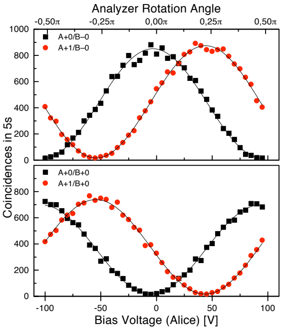


Figure: Four out of sixteen coincidence rates between various detection channels as functions of bias voltage (analyzer rotation angle) on Alice's modulator. A+1/B−0 for example are the coincidences between Alice's "+" detector with switch having been in position "1" and Bob's "−" detector with switch position "0". The difference in height can be explained by different efficiencies of the detectors.

[Figure from G. Weihs *et al.*, Phys. Rev. Lett. **81** 5039 (1998).]

Outline

Wigner functions and LHV models

- Is there a connection between other areas of physics and local hidden variable models?

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- Yes. For example, a surprising connection can be seen with Wigner functions.

Wigner functions and LHV models

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- Yes. For example, a surprising connection can be seen with Wigner functions.

- Wigner functions $W(x,p)$ are defined for a single bosonic model such that

$$\left\langle (x^m p^n)_{\mathrm{sym}} \right\rangle = \int x^m p^n W(x,p) dxdp.$$

- The Wigner function is a quasi-probability distribution. That is

$$\int W(x,p) dxdp = 1,$$

but $W(x,p)$ can also be negative.

- If $W(x,p) \geq 0$ for all x and p, then it is a real probability distribution.

Wigner functions and LHV models II

- If $W(x, p) \geq 0$ for all x and p, then it behaves as if there were a joint probability of the type

  $$P(x_0 \leq x \leq x_0{+}dx, p_0 \leq p \leq p_0{+}dp) = W(x_0, p_0)dxdp \text{ (Wrong.)}$$

- In reality, this is not the case. If we measure x, to ask about the value of p does not make sense, and vice versa.

- If $W(x, p) \geq 0$ for all x and p, then there is something like a LHV model for x and p. (However, note that they are measured on the same particle.)

Outline

What is new in a quantum system compared to its classical counterpart?

- Let us compare a classical bit to a quantum bit (qubit)
  - A classical bit is either in state "0" or in state "1".
  - A qubit (two-state system) can be in a superposition of the two.

$$|\Psi\rangle = c_0|0\rangle + c_1|1\rangle,$$

  where c_0 and c_1 are complex numbers. It is usual to use the shorthand notation, write

$$|\Psi\rangle = \begin{pmatrix} c_0 \\ c_1 \end{pmatrix},$$

  and call $|\Psi\rangle$ the state vector.
- To describe a quantum system one needs more degrees of freedom.

Two qubits

- Let us consider a two-qubit system. Naively, one could think that

$$|\Psi_1\rangle = c_0|0\rangle + c_1|1\rangle,$$

$$|\Psi_2\rangle = d_0|0\rangle + d_1|1\rangle,$$

- However, the correct picture is that the two-qubit system is described by

$$|\Psi_{12}\rangle = K_0|00\rangle + K_1|01\rangle + K_2|10\rangle + K_3|11\rangle$$

where K's are complex constants.

- Note that the number of the degrees of freedom in the second case is larger.

Two qubits II

- The naive picture assumes that the two systems are in a certain quantum state independently of the other system.
- There are quantum states like that, for example,

$$|\Psi_1\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle,$$
$$|\Psi_2\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle,$$

corresponds to

$$\begin{aligned}|\Psi_{12}\rangle &= |\Psi_1\rangle \otimes |\Psi_2\rangle \\ &= \Big(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\Big) \otimes \Big(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\Big) \\ &= \frac{1}{4}\Big(|00\rangle + |01\rangle + |10\rangle + |11\rangle\Big).\end{aligned}$$

These are the product states that are examples of separable states.

- States that cannot be written in this product form are the entangled states.

Mixed states

- So far we were talking about pure quantum states.
- In a real experiment quantum states are mixed. Such states can be described by a density matrix

$$\rho = \sum_k p_k |\Psi_k\rangle (|\Psi_k\rangle)^\dagger = \sum_k p_k |\Psi_k\rangle\langle\Psi_k|,$$

where $\sum_k p_k = 1$ and $p_k \geq 0$.

Definition

A mixed state is separable if it can be written as the convex combination of product states

$$\rho = \sum_k p_k \rho_k^{(1)} \otimes \rho_k^{(2)}.$$

Otherwise the state is entangled. [R. Werner, Phys. Rev. A 1989.]

Convexity

- Properties of density matrices

$$\begin{aligned} \rho &= \rho^\dagger, \\ \mathrm{Tr}(\rho) &= 1, \\ \rho &\geq 0. \end{aligned}$$

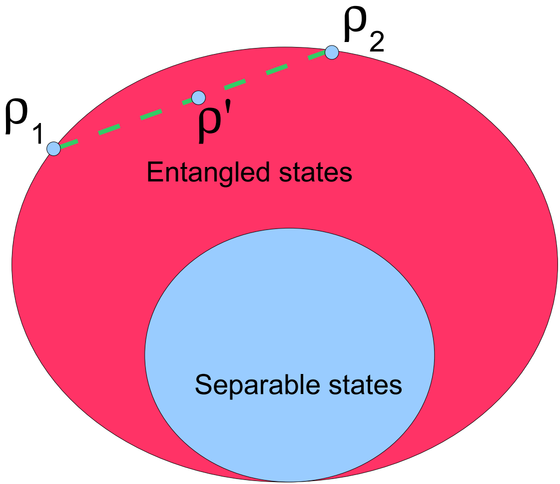
- Mixing two systems:

$$\rho' = p\rho_1 + (1 - p)\rho_2.$$

- The set of density matrices is convex. If ρ_1 and ρ_2 are density matrices then ρ' is also a density matrix.
- The set of density matrices corresponding to separable states is also convex. If ρ_1 and ρ_2 are separable density matrices then ρ' is also a separable density matrix.

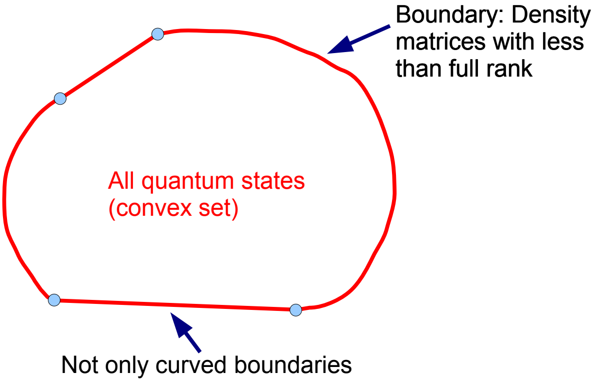
Convex sets

- Now, if we use the elements of the density matrix as coordinate axes, we can draw the following picture:



Convex sets II

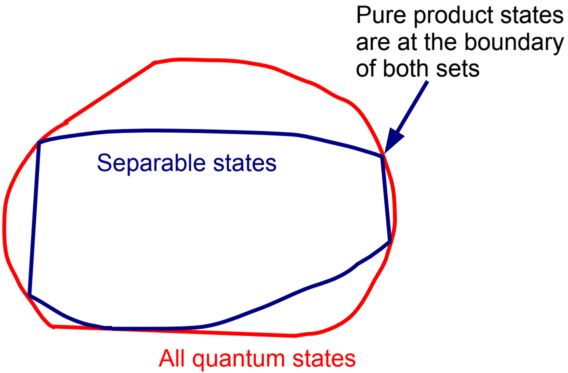
- A more correct figure is the following:



- Non-full rank density matrices have a zero eigenvalue. For a two-state system, the pure states are on the boundary.

Convex sets III

- A more correct figure for both sets is the following:



Entanglement cannot be created locally

- Remember: The definition of a separable state is

$$\rho = \sum_k p_k \rho_k^{(1)} \otimes \rho_k^{(2)}.$$

Definition

Local Operation and Classical Communications (LOCC):

- Single-party unitaries,
- Single party von Neumann measurements,
- Single party POVM measurements,
- We are even allowed to carry out measurement on party 1 and depending on the result, perform some other operation on party 2 ("Classical Communication").

LOCC and entanglement

It is not possible to create entangled states from separable states, with LOCC.

Entanglement is a resource

- In short: Starting from a separable state, we cannot create entanglement without real two-party quantum dynamics.
- In some cases such dynamics is impossible. For example, if we talk about particles very far away from each other.
- Then, we can transform entangled states to other entangled states, but cannot start from separable states and obtain entangled states.
- Thus, entangled states are a resource in this case.

Why is entanglement important?

- Can be used for quantum information processing protocols, quantum teleportation or quantum cryptography.
- Important for quantum algorithms such as prime factoring or search.
- Can also be used in quantum metrology (i.e., atomic clocks).
- Entanglement is a natural goal for experiments.

Entanglement distillation

- From many entangled particle pairs we want to create fewer strongly entangled pairs with LOCC.

Entanglement of distillation and entanglement of formation

- For the case of two-qibit, typically the aim is to create singlets that are maximally entangled states.

$$|\Psi_{\text{singlet}}\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle).$$

Definition

The entanglement of distillation a quantum state is characterized by determining, how many singlets can be distilled from it by LOCC.

- One can ask the opposite question. How many singlets are needed to construct a quantum state. item

Definition

The entanglement of formation a quantum state is characterized by determining, how many singlets are needed to form the state by LOCC.

Entanglement of distillation and entanglement of formation II

- For pure states, the entanglement of formation equals the von Neumann entropy of the *reduced* state

$$E_F = -\mathrm{Tr}(\rho_1 \log_2 \rho_1).$$

- In general, the entanglement of formation is not smaller than the entanglement of distillation

$$E_F \geq E_D.$$

Definition

There are entangled quantum states, that need singlets to create them, but no singlets can be distilled by LOCC. These are called bound entangled states.

Entanglement criteria

- How to decide whether a quantum state with a given density matrix is entangled?
- For pure states it is simple. A pure state is entangled if it is not a product state.
- A mixed state is entangled if it cannot be written as

$$\rho = \sum_k p_k \rho_k^{(1)} \otimes \rho_k^{(2)}.$$

But how can we find out whether a quantum state can be decomposed like that?

The positivity of the partial transpose (PPT) criterion

Definition

For a separable state ϱ, the partial transpose is always positive semidefinite

$$\varrho^{T1} \geq 0.$$

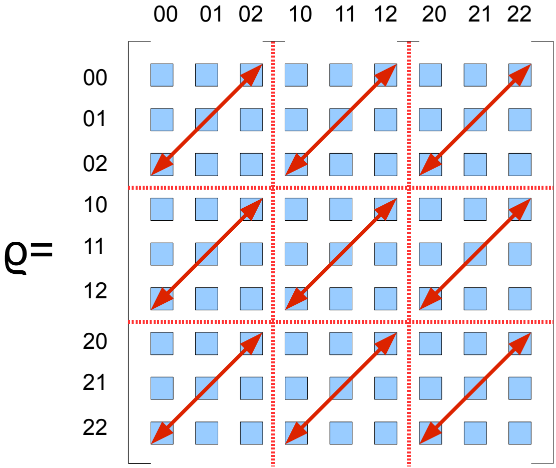
If a state does not have a positive semidefinite partial transpose, then it is entangled. [A. Peres, PRL 1996; Horodecki *et al.*, PLA 1997.]

- Partial transpose means transposing according to one of the two subsystems.
- For separable states

$$(T \otimes \mathbb{1})\varrho = \varrho^{T1} = \sum_k p_k (\varrho_k^{(1)})^T \otimes \varrho_k^{(2)} \geq 0.$$

The positivity of the partial transpose (PPT) criterion II

- How to obtain the partial transpose of a general density matrix? Example: 3×3 case.



PPT entangled states are bound entangled

- The PPT criterion detects all entangled states for 2×2 and 2×3 systems.
- For larger systems, it does not detect all entangled states. E.g., for 3x3 systems there are PPT entangled states.
- It can be shown that no entanglement can be distilled from them, thus they are bound entangled.

The Computable Cross Norm-Realignment Criterion

Definition

Let us consider a quantum state ϱ, with a Schmidt decomposition

$$\varrho = \sum_k \lambda_k G_k^{(A)} \otimes G_k^{(B)},$$

where $\mathrm{Tr}(G_m^{(l)} G_n^{(l)}) = \delta_{mn}$ and $\lambda_k \geq 0$. If ϱ is separable then $\sum_k \lambda_k \leq 1$.

[O. Rudolph, Quant. Inf. Proc. 2005; K. Chen and L.A. Wu, Quant. Inf. Comp. 2003.]

- *Proof.* For product states the Schmidt decomposition of the density matrix is

$$\varrho_{\mathrm{product}} = |\Psi_A\rangle\langle\Psi_A| \otimes |\Psi_B\rangle\langle\Psi_B|.$$

The Computable Cross Norm-Realignment Criterion II

- For mixed states, we have to use that

$$\sum_k \lambda_k$$

defines a norm for quantum states that is convex.

- Other definition of CCNR criterion is based on a "realignment" of the density matrix.

Entanglement detection with uncertainty relations

- We have a bipartite system and the following operators
  - A_1 and B_1 act on the first party.
  - A_2 and B_2 act on the second party.
- If for quantum states

$$(\Delta A_k)^2 + (\Delta B_k)^2 \geq c,$$

- then for separable states we have

$$(\Delta A_1 + A_2)^2 + (\Delta B_1 + B_2)^2 \geq 2c.$$

[H.F. Hofmann and S. Takeuchi PRA 2003; O. Gühne, PRL 2004.]

Entanglement detection with uncertainty relations II

- Proof: For product states

$$|\Psi\rangle = |\Psi_1\rangle \otimes |\Psi_2\rangle$$

we have

$$(\Delta(A_1 + A_2))^2 + (\Delta(B_1 + B_2))^2 = (\Delta A_1)^2_{\Psi_1} + (\Delta B_1)^2_{\Psi_1} + (\Delta A_2)^2_{\Psi_2} + (\Delta B_2)^2_{\Psi_2} \geq 2c.$$

- Separable states are mixtures of pure states. Due to convexity this bound is also valid for separable states.

- Simple example for two-mode systems

$$(\Delta(x_1 + x_2))^2 + (\Delta(p_1 - p_2))^2 \geq 2.$$

Entanglement detection with a single nonlocal measurement: Entanglement witnesses

- An operator W is an entanglement witness if $\langle W \rangle = \mathrm{Tr}(W\rho) < 0$ only for entangled states.

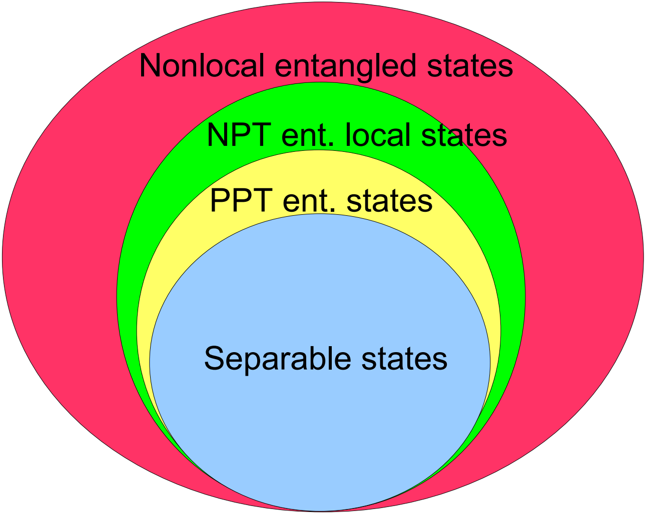
[Horodecki et al., Phys. Lett. A 223, 8 (1996); Terhal, quant-ph/9810091; Lewenstein, Phys. Rev. A 62, 052310 (2000).]

Entanglement vs. Nonlocality

- All states that violate a Bell inequality are entangled.
- Equivalently, separable states do not violate any Bell inequality.
- However, there are entangled states that do not violate any Bell inequality. [R.F. Werner, PRA 1989.]
- It is conjectured by Peres that every PPT state is local. Sofar no counterexamples have been found.

Entanglement vs. Nonlocality

- The relations of the various convex sets look like as follows



Outline

Many-body quantum systems

- An N-qubit mixed state is separable if it can be written as

$$\rho = \sum_k p_k \rho_k^{(1)} \otimes \rho_k^{(2)} \otimes \rho_k^{(3)} \otimes ... \otimes \rho_k^{(N)}.$$

  Otherwise the state is entangled.

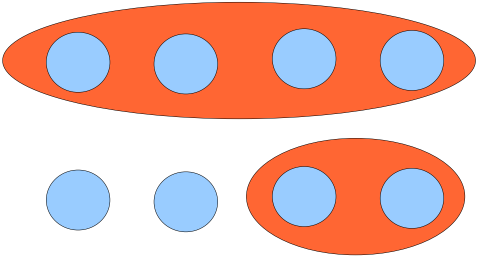
- A bipartite quantum state is either separable or entangled. The multipartite case is more complicated.
- We have to distinguish between quantum states in which only some of the qubits are entangled from those in which all the qubits are entangled.
- Biseparable states are the states that might be entangled but they are separable with respect to some partition. States that are not biseparable are called genuine multipartite entangled.

Genuine multipartite entanglement

- Let us see two entangled states of four qubits:

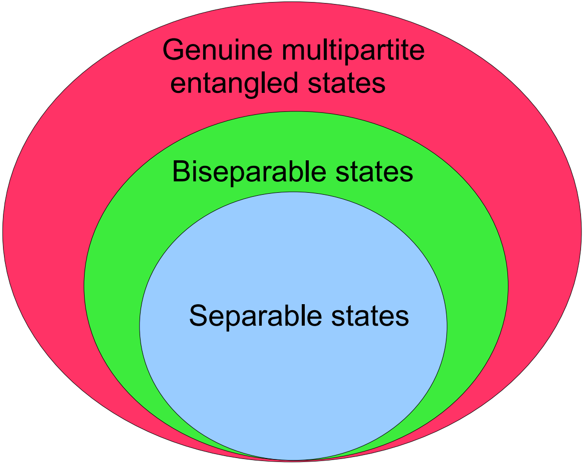
$$
\begin{aligned}
|GHZ_4\rangle &= \frac{1}{\sqrt{2}}(|0000\rangle + |1111\rangle), \\
|\Psi_B\rangle &= \frac{1}{\sqrt{2}}(|0000\rangle + |0011\rangle) = \frac{1}{\sqrt{2}}|00\rangle \otimes (|00\rangle + |11\rangle).
\end{aligned}
$$

- The first state is genuine multipartite entangled, the second state is biseparable.



Convex sets for the multi-qubit case

- The idea also works for the multi-qubit case: A state is biseparable if it can be composed by mixing pure bisparable states.



Outline

Detection of entanglement

- Many quantum engineering/quantum control experiments have two main steps:
  - Creation of an entangled quantum state,
  - Detection its entanglement.
- Thus entanglement detection is one of the most important subjects in this field.
- Examples of quantum control experiments:
  - Nuclear spin of atoms in a molecule (NMR): ≤ 10 qubits
  - Parametric down-conversion and post-selection: ≤ 10 qubits
  - Trapped ion experiments: ≤ 8 qubits

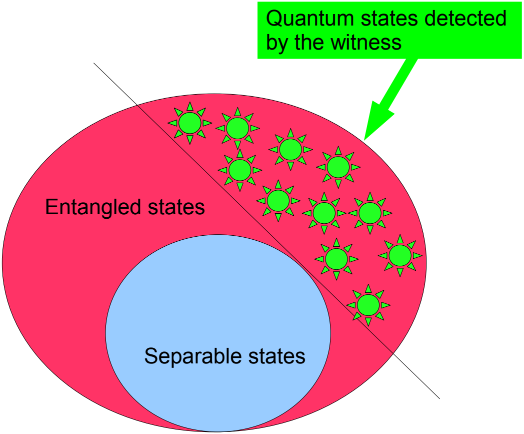
Entanglement detection with tomography

- Determine the density matrix and apply an entanglement criterion.
- For N qubits the density matrix has $2^N \times 2^N$ complex elements, and has $2^{2N} - 1$ real degrees of freedom.
  - 10 qubits $\rightarrow \sim 1$ million measuremets
  - 20 qubits $\rightarrow \sim 10^{12}$ measuremets
- Surprise: Above modest system sizes full tomography is not possible. One has to find methods for entanglement detection that are feasible even without knowing the quantum state.

Entanglement detection with a single nonlocal measurement: Entanglement witnesses

- An operator W is an entanglement witness if $\langle W \rangle = \mathrm{Tr}(W\rho) < 0$ only for entangled states.

[Horodecki et al., Phys. Lett. A 223, 8 (1996); Terhal, quant-ph/9810091; Lewenstein, Phys. Rev. A 62, 052310 (2000).]



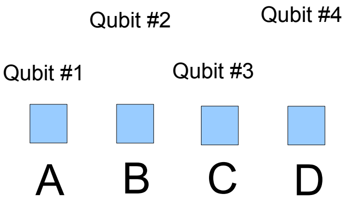
Entanglement detection with local measurements

- Example:

$$W_{GHZ} := \frac{1}{2}\mathbb{1} - |GHZ\rangle\langle GHZ|$$

  is a witness, where $|GHZ\rangle := (|000..00\rangle + |111..11\rangle)/\sqrt{2}$.
  W_{GHZ} detects entanglement in the vicinity of GHZ states.

- Problem: Only local measurements are possible. With local measurements, operators of the type $\langle A^{(1)}B^{(2)}C^{(3)}C^{(4)}\rangle$ can be measured.



Entanglement detection with local measurements II

- All operators must be decomposed into the sum of locally measurable terms and these terms must be measured individually.
- For example,

$$
\begin{aligned}
|GHZ_3\rangle\langle GHZ_3| &= \frac{1}{8}(\mathbb{1} + \sigma_z^{(1)}\sigma_z^{(2)} + \sigma_z^{(1)}\sigma_z^{(3)} + \sigma_z^{(2)}\sigma_z^{(3)}) \\
&+ \frac{1}{4}\sigma_x^{(1)}\sigma_x^{(2)}\sigma_x^{(3)}) \\
&- \frac{1}{16}(\sigma_x^{(1)} + \sigma_y^{(1)})(\sigma_x^{(2)} + \sigma_y^{(2)})(\sigma_x^{(3)} + \sigma_y^{(3)}) \\
&- \frac{1}{16}(\sigma_x^{(1)} - \sigma_y^{(1)})(\sigma_x^{(2)} - \sigma_y^{(2)})(\sigma_x^{(3)} - \sigma_y^{(3)}).
\end{aligned}
$$

[O. Gühne és P. Hyllus, Int. J. Theor. Phys. 42, 1001-1013 (2003). M. Bourennane et al., Phys. Rev. Lett. 92 087902 (2004).]

- As N increases, the number of terms increases exponentially.

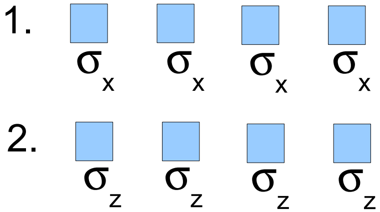
Solution: Entanglement witnesses designed for detection with few measurements

- Alternative witness with easy decomposition

$$W'_{GHZ} := 3\mathbb{1} - 2\left[\frac{\sigma_x^{(1)}\sigma_x^{(2)}\cdots\sigma_x^{(N-1)}\sigma_x^{(N)}}{2} + \prod_{k=2}^{N}\frac{\sigma_z^{(k)}\sigma_z^{(k+1)} + \mathbb{1}}{2}\right].$$

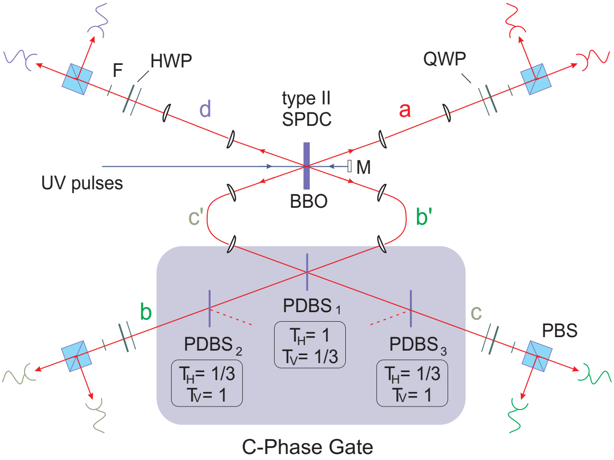
Note that $W'_{GHZ} \geq 2W_{GHZ}$. [GT and O. Gühne, Phys. Rev. Lett. 94, 060501 (2005).]

- The number of local measurements does not increases with N.



Example: An experiment

- Creation of a four-qubit cluster state with photons and its detection [Figure from Kiesel, C. Schmid, U. Weber, GT, O. Gühne, R. Ursin, and H. Weinfurter, Phys. Rev. Lett. 95, 210502; See also GT and O. Gühne, Phys. Rev. Lett. 94, 060501 (2005).]



Outline

1. Introduction
2. Nonlocality, Bell inequalities and hidden variable models
   1. Einstein-Podolsky-Rosen (EPR) paradox
   2. Bipartite nonlocality
   3. Multipartite nonlocality
   4. Connection to other areas of physics: Wigner functions
3. Entanglement, entanglement witnesses
   1. Bipartite quantum entanglement
   2. Many-body entanglement
   3. Entanglement detection in a system of few particles
   4. Entanglement detection in many-body experiments

Very many particles

- Typically we cannot address the particles individually.
- Expected to occur often in future experiments.
- For spin-$\frac{1}{2}$ particles, we can measure the collective angular momentum operators:

$$J_l := \frac{1}{2}\sum_{k=1}^{N}\sigma_l^{(k)},$$

  where $l = x, y, z$ and $\sigma_l^{(k)}$ a Pauli spin matrices.
- We can also measure the $(\Delta J_l)^2 := \langle J_l^2\rangle - \langle J_l\rangle^2$ variances.

Spin squeezing I.

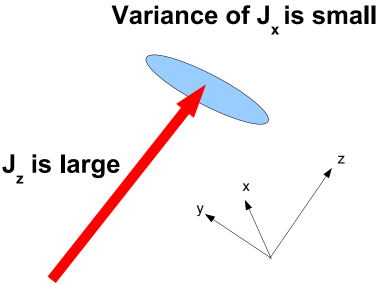
- Uncertainty relation for the spin coordinates:

$$(\Delta J_x)^2(\Delta J_y)^2 \geq \frac{1}{4}|\langle J_z\rangle|^2.$$

- If $(\Delta J_x)^2$ is smaller than the standard quantum limit $\frac{1}{2}|\langle J_z\rangle|$ then the state is called spin squeezed.

[M. Kitagawa and M. Ueda, Phys. Rev. A 47, 5138 (1993).]

- Application: Quantum metrology.



Spin squeezing II.

- Spin squeezing experiment with 10^7 atoms: [J. Hald, J. L. Sørensen, C. Schori, and E. S. Polzik, Phys. Rev. Lett. 83, 1319 (1999)]

- Spin squeezing criterion for the detection of quantum entanglement

$$\frac{(\Delta J_x)^2}{\langle J_y\rangle^2 + \langle J_z\rangle^2} \geq \frac{1}{N}.$$

If a quantum state violates this criterion then it is entangled.

[A. Sørensen *et al.*, Nature **409**, 63 (2001).]

Generalized spin squeezing criteria

- Criterion 1. For separable states we have

$$\langle J_x^2\rangle + \langle J_y^2\rangle \le \frac{N}{4}(N+1).$$

  This detects entangled states close to symmetric Dicke states $\langle J_z\rangle = 0$. E.g., for $N = 4$-re this state is $\frac{1}{\sqrt{6}}(|0011\rangle + |0101\rangle + |1001\rangle + |0110\rangle + |1010\rangle + |1100\rangle)$.

  [GT, J. Opt. Soc. Am. B **24**, 275 (2007); N. Kiesel *et al.*, Phys. Rev. Lett. 98, 063604 (2007).]

- Criterion 2. For separable states

$$(\Delta J_x)^2 + (\Delta J_y)^2 + (\Delta J_z)^2 \ge N/2.$$

  The left hand side is zero for the ground state of a Heisenberg chain. [GT, Phys. Rev. A **69**, 052327 (2004).]

- Criterion 3. For symmetric separable states $1 - 4\langle J_m\rangle^2/N^2 \le 4(\Delta J_m)^2/N$. [J. Korbicz *et al.* Phys. Rev. Lett. **95**, 120502 (2005).]

- How could we find all such criteria?

Complete set of generalized spin squeezing inequalities

- Let us assume that for a system we know only

$$\mathbf{J} := (\langle J_x\rangle, \langle J_y\rangle, \langle J_z\rangle),$$
$$\mathbf{K} := (\langle J_x^2\rangle, \langle J_y^2\rangle, \langle J_z^2\rangle).$$

- Then any state violating the following inequalities is entangled

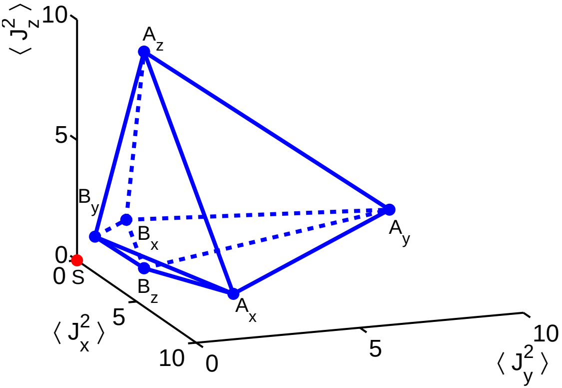
$$\begin{aligned}
\langle J_x^2\rangle + \langle J_y^2\rangle + \langle J_z^2\rangle &\le N(N+2)/4, \\
(\Delta J_x)^2 + (\Delta J_y)^2 + (\Delta J_z)^2 &\ge N/2, \\
\langle J_k^2\rangle + \langle J_l^2\rangle - N/2 &\le (N-1)(\Delta J_m)^2, \\
(N-1)\left[(\Delta J_k)^2 + (\Delta J_l)^2\right] &\ge \langle J_m^2\rangle + N(N-2)/4,
\end{aligned}$$

where k, l, m takes all the possible permutations of x, y, z.

[GT, C. Knapp, O. Gühne, és H.J. Briegel, Phys. Rev. Lett. 2007.]

The polytope

- The previous inequalities, *for fixed* $\langle J_{x/y/z}\rangle$, describe a polytope in the $\langle J^2_{x/y/z}\rangle$ space.
- Separable states correspond to points inside the polytope. Note: Convexity comes up again!
- For $\langle \mathbf{J}\rangle = 0$ and $N = 6$ the polytope is the following:



Conclusions

- We discussed Bell inequalities and local hidden variable models
- We discussed separability and entanglement.
- We also discussed entanglement criteria and entanglement detection in experiments.

For further information please see my home page:

http://optics.szfki.kfki.hu/~toth

and the review

O. Gühne and G. Tóth, Physics Reports 474, 1-75 (2009).

*** THANK YOU ***