Quantum entanglement and its detection with few measurements

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Outline

1. Introduction
2. Bipartite quantum entanglement
3. Many-body entanglement
4. Entanglement detection in a system of few particles
5. Entanglement detection in many-body experiments

Outline

Introduction

- Due to the rapid development of quantum engineering and quantum control, it is now possible to carry out experiments in which many-body quantum systems undergo coherent dynamics.
  - Few particles (< 10), creation of interesting quantum states in various physical systems, such as trapped ions, photonic systems, or molecules controlled by nuclear magnetic resonance (NMR).
  - Large scale (e.g., 10^5 particles) systems, for example, optical lattices of cold two-state atoms and cold atomic clouds.
- These experiments are possible due to novel technologies developed in the last ten years.

Outline

What in new a quantum system compared to its classical counterpart?

- Let us compare a classical bit to a quantum bit (qubit)
  - A classical bit is either in state "0" or in state "1".
  - A qubit (two-state system) can be in a superposition of the two.

$$|\Psi\rangle = c_0|0\rangle + c_1|1\rangle,$$

  where c_0 and c_1 are complex numbers. It is usual to use the shorthand notation, write

$$|\Psi\rangle = \begin{pmatrix} c_0 \\ c_1 \end{pmatrix},$$

  and call $|\Psi\rangle$ the state vector.

- To describe a quantum system one needs more degrees of freedom.

Two qubits

- Let us consider a two-qubit system. Naively, one could think that

$$|\Psi_1\rangle = c_0|0\rangle + c_1|1\rangle,$$

$$|\Psi_2\rangle = d_0|0\rangle + d_1|1\rangle,$$

- However, the correct picture is that the two-qubit system is described by

$$|\Psi_{12}\rangle = K_0|00\rangle + K_1|01\rangle + K_2|10\rangle + K_3|11\rangle$$

  where K's are complex constants.

- Note that the number of the degrees of freedom in the second case is larger.

Two qubits II

- The naive picture assumes that the two systems are in a certain quantum state independently of the other system.
- There are quantum states like that, for example,

$$|\Psi_1\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle,$$
$$|\Psi_2\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle,$$

corresponds to

$$\begin{aligned} |\Psi_{12}\rangle &= |\Psi_1\rangle \otimes |\Psi_2\rangle \\ &= \left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right) \otimes \left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right) \\ &= \frac{1}{4}(|00\rangle + |01\rangle + |10\rangle + |11\rangle). \end{aligned}$$

These are the product states that are examples of separable states.

- States that cannot be written in this product form are the entangled states.

Mixed states

- So far we were talking about pure quantum states.
- In a real experiment quantum states are mixed. Such states can be described by a density matrix

$$\rho = \sum_k p_k |\Psi_k\rangle (|\Psi_k\rangle)^\dagger = \sum_k p_k |\Psi_k\rangle\langle\Psi_k|,$$

where $\sum_k p_k = 1$ and $p_k \geq 0$.

- A mixed state is separable if it can be written as the convex combination of product states

$$\rho = \sum_k p_k \rho_k^{(1)} \otimes \rho_k^{(2)}.$$

Otherwise the state is entangled. [R. Werner, Phys. Rev. A 1989.]

Convexity

- Properties of density matrices

$$\begin{aligned} \rho &= \rho^\dagger, \\ \mathrm{Tr}(\rho) &= 1, \\ \rho &\geq 0. \end{aligned}$$

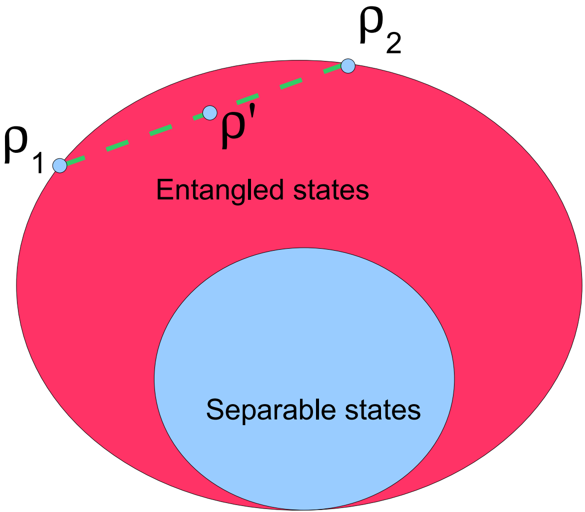
- Mixing two systems:

$$\rho' = p\rho_1 + (1-p)\rho_2. \qquad (1)$$

- The set of density matrices is convex. If ρ_1 and ρ_2 are density matrices then ρ' is also a density matrix.
- The set of density matrices corresponding to separable states is also convex. If ρ_1 and ρ_2 are separable density matrices then ρ' is also a separable density matrix.

Convex sets

- Now, if we use the elements of the density matrix as coordinate axes, we can draw the following picture:



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Many-body quantum systems

- An N-qubit mixed state is separable if it can be written as

$$\rho = \sum_k p_k \rho_k^{(1)} \otimes \rho_k^{(2)} \otimes \rho_k^{(3)} \otimes ... \otimes \rho_k^{(N)}.$$

  Otherwise the state is entangled.

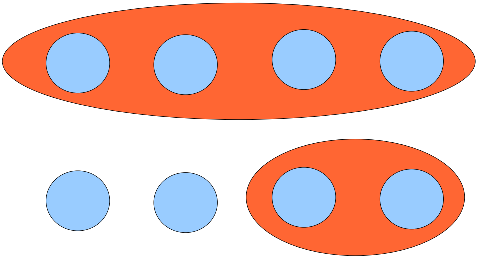
- A bipartite quantum state is either separable or entangled. The multipartite case is more complicated.
- We have to distinguish between quantum states in which only some of the qubits are entangled from those in which all the qubits are entangled.
- Biseparable states are the states that might be entangled but they are separable with respect to some partition. States that are not biseparable are called genuine multipartite entangled.

Genuine multipartite entanglement

- Let us see two entangled states of four qubits:

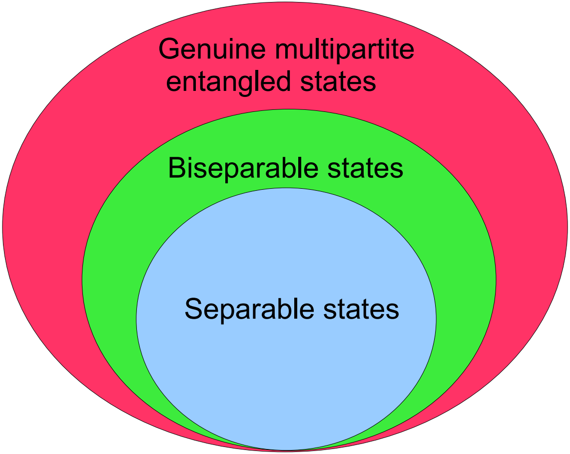
$$|GHZ_4\rangle = \frac{1}{\sqrt{2}}(|0000\rangle + |1111\rangle),$$
$$|\Psi_B\rangle = \frac{1}{\sqrt{2}}(|0000\rangle + |0011\rangle) = \frac{1}{\sqrt{2}}|00\rangle \otimes (|00\rangle + |11\rangle).$$

- The first state is genuine multipartite entangled, the second state is biseparable.



Convex sets for the multi-qubit case

- The idea also works for the multi-qubit case: A state is biseparable if it can be composed by mixing pure bisparable states.



Why is entanglement important?

- Can be used for quantum information processing protocols, quantum teleportation or quantum cryptography.
- Important for quantum algorithms such as prime factoring or search.
- Can also be used in quantum metrology (i.e., atomic clocks).
- Entanglement is a natural goal for experiments.

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Detection of entanglement

- Many quantum engineering/quantum control experiments have two main steps:
  - Creation of an entangled quantum state,
  - Detection its entanglement.
- Thus entanglement detection is one of the most important subjects in this field.
- Examples of quantum control experiments:
  - Nuclear spin of atoms in a molecule (NMR): $\leq$ 10 qubits
  - Parametric down-conversion and post-selection: $\leq$ 6 qubits
  - Trapped ion experiments: $\leq$ 8 qubits

Entanglement detection with tomography

- Determine the density matrix and apply an entanglement crierion.

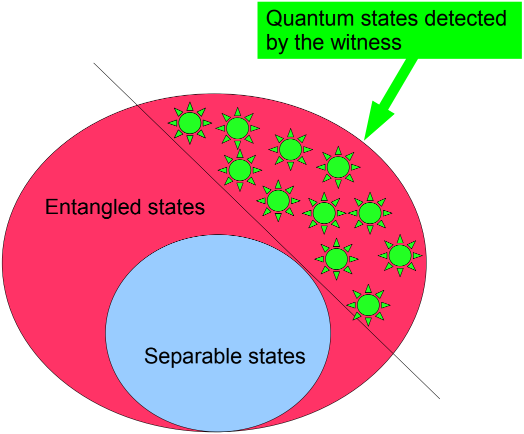
- For N qubits the density matrix has $2^N \times 2^N$ complex elements, and has $2^{2N} - 1$ real degrees of freedom.
  - 10 qubits $\rightarrow \sim$ 1 million measuremets
  - 20 qubits $\rightarrow \sim 10^{12}$ measuremets

- Surprise: Above modest system sizes full tomography is not possible. One has to find methods for entanglement detection that are feasible even without knowing the quantum state.

Entanglement detection with a single nonlocal measurement: Entanglement witnesses

- An operator W is an entanglement witness if $\langle W \rangle = \mathrm{Tr}(W\rho) < 0$ only for entangled states.

[Horodecki et al., Phys. Lett. A 223, 8 (1996); Terhal, quant-ph/9810091; Lewenstein, Phys. Rev. A 62, 052310 (2000).]



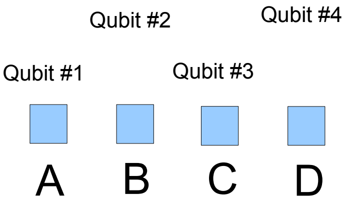
Entanglement detection with local measurements

- Example:

$$W_{GHZ} := \frac{1}{2}\mathbb{1} - |GHZ\rangle\langle GHZ|$$

  is a witness, where $|GHZ\rangle := (|000..00\rangle + |111..11\rangle)/\sqrt{2}$. W_{GHZ} detects entanglement in the vicinity of GHZ states.

- Problem: Only local measurements are possible. With local measurements, operators of the type $\langle A^{(1)}B^{(2)}C^{(3)}C^{(4)}\rangle$ can be measured.



Entanglement detection with local measurements II

- All operators must be decomposed into the sum of locally measurable terms and these terms must be measured individually.
- For example,

$$
\begin{aligned}
|GHZ_3\rangle\langle GHZ_3| &= \frac{1}{8}(5\mathbb{1} - \sigma_z^{(1)}\sigma_z^{(2)} - \sigma_z^{(1)}\sigma_z^{(3)} - \sigma_z^{(2)}\sigma_z^{(3)} \\
&- 2\sigma_x^{(1)}\sigma_x^{(2)}\sigma_x^{(3)}) \\
&+ \frac{1}{4}(\sigma_x^{(1)} + \sigma_y^{(1)})(\sigma_x^{(2)} + \sigma_y^{(2)})(\sigma_x^{(3)} + \sigma_y^{(3)}) \\
&+ \frac{1}{4}(\sigma_x^{(1)} - \sigma_y^{(1)})(\sigma_x^{(2)} - \sigma_y^{(2)})(\sigma_x^{(3)} - \sigma_y^{(3)}).
\end{aligned}
$$

[O. Gühne és P. Hyllus, Int. J. Theor. Phys. 42, 1001-1013 (2003). M. Bourennane et al., Phys. Rev. Lett. 92 087902 (2004).]

- As N increases, the number of terms increases exponentially.

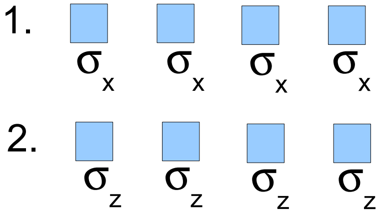
Solution: Entanglement witnesses designed for detection with few measurements

- Alternative witness with easy decomposition

$$W'_{GHZ} := 3\mathbb{1} - 2\left[\frac{\sigma_x^{(1)}\sigma_x^{(2)}\cdots\sigma_x^{(N-1)}\sigma_x^{(N)}\mathbb{1}}{2} + \prod_{k=2}^{N}\frac{\sigma_z^{(k)}\sigma_z^{(k+1)} + \mathbb{1}}{2}\right].$$

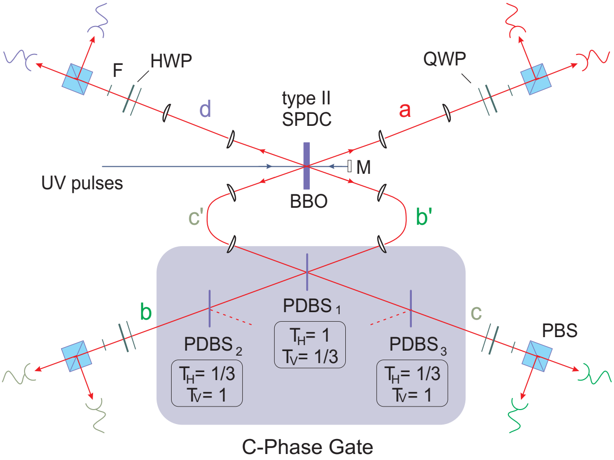
Note that $W'_{GHZ} \geq 2W_{GHZ}$. [GT and O. Gühne, Phys. Rev. Lett. 94, 060501 (2005).]

- The number of local measurements does not increases with N.



Example: An experiment

- Creation of a four-qubit cluster state with photons and its detection [Figure from Kiesel, C. Schmid, U. Weber, GT, O. Gühne, R. Ursin, and H. Weinfurter, Phys. Rev. Lett. 95, 210502; See also GT and O. Gühne, Phys. Rev. Lett. 94, 060501 (2005).]



Outline

Very many particles

- Typically we cannot address the particles individually.
- Expected to occur often in future experiments.
- For spin-$\frac{1}{2}$ particles, we can measure the collective angular momentum operators:

$$J_l := \frac{1}{2} \sum_{k=1}^{N} \sigma_l^{(k)},$$

where $l = x, y, z$ and $\sigma_l^{(k)}$ a Pauli spin matrices.

- We can also measure the $(\Delta J_l)^2 := \langle J_l^2 \rangle - \langle J_l \rangle^2$ variances.

Spin squeezing I.

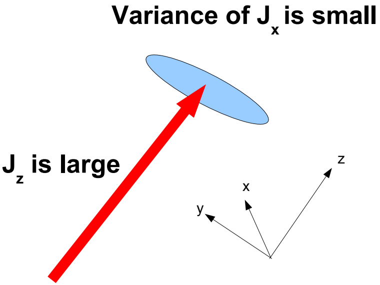
- Uncertainty relation for the spin coordinates:

$$(\Delta J_x)^2(\Delta J_y)^2 \geq \frac{1}{4}|\langle J_z \rangle|^2.$$

- If $(\Delta J_x)^2$ is smaller than the standard quantum limit $\frac{1}{2}|\langle J_z \rangle|$ then the state is called spin squeezed.

  [M. Kitagawa and M. Ueda, Phys. Rev. A 47, 5138 (1993).]

- Application: Quantum metrology.



Spin squeezing II.

- Spin squeezing experiment with 10^7 atoms: [J. Hald, J. L. Sørensen, C. Schori, and E. S. Polzik, Phys. Rev. Lett. 83, 1319 (1999)]

- Spin squeezing criterion for the detection of quantum entanglement

$$\frac{(\Delta J_x)^2}{\langle J_y\rangle^2 + \langle J_z\rangle^2} \geq \frac{1}{N}.$$

If a quantum state violates this criterion then it is entangled.

[A. Sørensen *et al.*, Nature **409**, 63 (2001).]

Generalized spin squeezing criteria

- Criterion 1. For separable states we have

$$\langle J_x^2 \rangle + \langle J_y^2 \rangle \leq \frac{N}{4}(N+1).$$

This detects entangled states close to symmetric Dicke states $\langle J_z \rangle = 0$. E.g., for $N = 4$-re this state is $\frac{1}{\sqrt{6}}(|0011\rangle + |0101\rangle + |1001\rangle + |0110\rangle + |1010\rangle + |1100\rangle)$.
[GT, J. Opt. Soc. Am. B **24**, 275 (2007); N. Kiesel *et al.*, Phys. Rev. Lett. 98, 063604 (2007).]

- Criterion 2. For separable states

$$(\Delta J_x)^2 + (\Delta J_y)^2 + (\Delta J_z)^2 \geq N/2.$$

The left hand side is zero for the ground state of a Heisenberg chain. [GT, Phys. Rev. A **69**, 052327 (2004).]

- Criterion 3. For symmetric separable states $1 - 4\langle J_m \rangle^2/N^2 \leq 4(\Delta J_m)^2/N$. [J. Korbicz *et al.* Phys. Rev. Lett. **95**, 120502 (2005).]

- How could we find all such criteria?

Complete set of generalized spin squeezing inequalities

- Let us assume that for a system we know only

$$\mathbf{J} := (\langle J_x \rangle, \langle J_y \rangle, \langle J_z \rangle),$$
$$\mathbf{K} := (\langle J_x^2 \rangle, \langle J_y^2 \rangle, \langle J_z^2 \rangle).$$

- Then any state violating the following inequalities is entangled

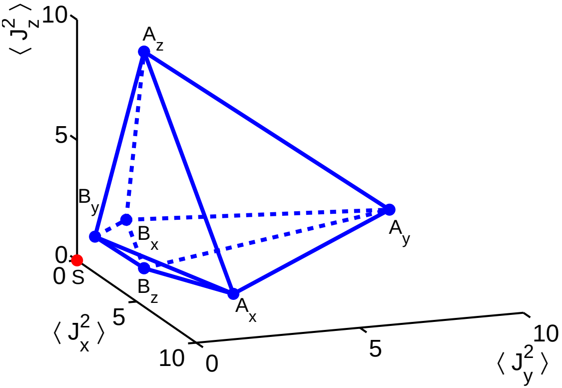
$$\begin{aligned} \langle J_x^2 \rangle + \langle J_y^2 \rangle + \langle J_z^2 \rangle &\leq N(N+2)/4, \\ (\Delta J_x)^2 + (\Delta J_y)^2 + (\Delta J_z)^2 &\geq N/2, \\ \langle J_k^2 \rangle + \langle J_l^2 \rangle - N/2 &\leq (N-1)(\Delta J_m)^2, \\ (N-1)\left[(\Delta J_k)^2 + (\Delta J_l)^2\right] &\geq \langle J_m^2 \rangle + N(N-2)/4, \end{aligned}$$

where k, l, m takes all the possible permutations of x, y, z.

[GT, C. Knapp, O. Gühne, és H.J. Briegel, Phys. Rev. Lett., in press; quant-ph/0702219.]

The polytope

- The previous inequalities, *for fixed* $\langle J_{x/y/z} \rangle$, describe a polytope in the $\langle J^2_{x/y/z} \rangle$ space.
- Separable states correspond to points inside the polytope. Note: Convexity comes up again!
- For $\langle \mathbf{J} \rangle = 0$ and $N = 6$ the polytope is the following:



Conclusions

- We discussed why entanglement is important for quantum information processing and also for quantum control experiments.
- We discussed entanglement detection in few particle systems. Only local measurements are possible.
- We also discussed entanglement detection in many particle systems. The particles cannot be addressed individually and only collective quantities can be measured. Generalized spin squeezing inequalities can be used for entanglement detection.

For further information please see my home page:
http://optics.szfki.kfki.hu/~toth

*** THANK YOU ***