Witnessing *Genuine* Many-qubit Entanglement with only Two Local Measurement Settings

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Outline

- Genuine multi-qubit entanglement
- Entanglement detection with entanglement witnesses
- Witness based on projectors
- Our proposal: witness with few local measurements (for GHZ & cluster states)
- Connection to Bell inequalities
- Entanglement detection with collective measurement

Genuine multi-qubit entanglement

- Genuine three-qubit entanglement

$$|000\rangle + |111\rangle$$

- Biseparable entanglement

$$|001\rangle + |111\rangle =$$

$$\left(|00\rangle + |11\rangle\right)|1\rangle$$

- A mixed entangled state is *biseparable* if it is the mixture of biseparabe states (of possibly different partitions).

Entanglement witnesses I

- Bell inequalities

  Classical: no knowledge of quantum mechanics is used to construct them.

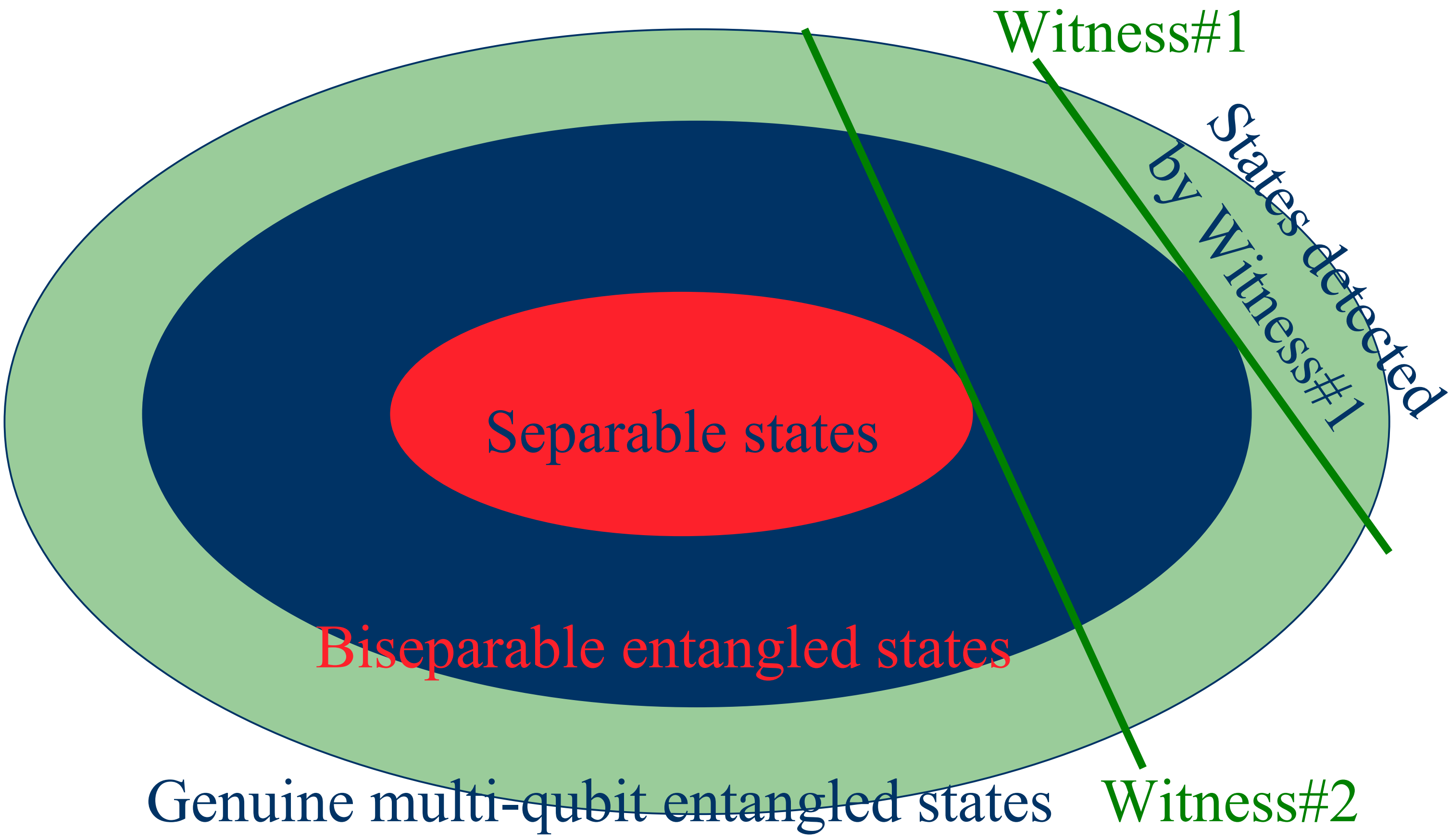
- Entanglement witnesses

  Knowledge of QM is used for constructing them.

Entanglement witnesses II

- Entanglement witnesses are observables which have
  - positive expectation values for separable states
  - negative expectation values for *some* entangled states.
- Witnesses can be constructed which detect entangled states close to a state chosen by us.
- Witnesses can be constructed which detect *only genuine multi-party* entanglement.

Entanglement witnesses III



Entanglement witnesses IV

- It is possible to construct witnesses for detecting entangled states close to a particular state with a projector. E.g.,

$$W_{GHZN}^{PROJ} = \frac{1}{2} \bullet 1 - |GHZ_N\rangle\langle GHZ_N|$$

detects N-qubit entangled states close to an N-qubit GHZ state.

Entanglement witnesses V

- So if

$$\left\langle W_{GHZN}^{PROJ} \right\rangle < 0$$

then the system is genuinely multi-qubit entangled.

- Question: how can we measure the witness operator?

Decomposing the witness

- For an experiment, the witness must be decomposed into locally measurable terms

$$W_{GHZ3}^{PROJ} = \frac{1}{8}(3\bullet 1 - \sigma_z^1\sigma_z^2 - \sigma_z^1\sigma_z^3 - \sigma_z^2\sigma_z^3 - 2\sigma_x^1\sigma_x^2\sigma_x^3)$$

$$+\frac{1}{4}\left(\sigma_x^1 + \sigma_y^1\right)\left(\sigma_x^2 + \sigma_y^2\right)\left(\sigma_x^3 + \sigma_y^3\right)$$

$$+\frac{1}{4}\left(\sigma_x^1 - \sigma_y^1\right)\left(\sigma_x^2 - \sigma_y^2\right)\left(\sigma_x^3 - \sigma_y^3\right)$$

- See O. Gühne, P. Hyllus, quant-ph/0301162; M. Bourennane et. al., PRL 92 087902 (2004).

Main topic of the talk: How can one decrease the number of local terms

- As the number of qubits increases, the number of local terms increases exponentially. Similar thing happens to Bell inequalities for the GHZ state.
- Q: How can we construct entanglement witnesses with few locally measurable terms?

Entanglement witnesses based on the stabilizer formalism

Stabilizer witnesses

- We propose new type of witnesses. E.g., for three-qubit GHZ states

$$W_{GHZ3} = \frac{3}{2} \bullet 1 - \sigma_x^1 \sigma_x^2 \sigma_x^3 - \frac{1}{2} \left[\sigma_z^1 \sigma_z^2 + \sigma_z^2 \sigma_z^3 + \sigma_z^1 \sigma_z^3 \right]$$

- All the three terms are +1 for the GHZ state.

Stabilizer witnesses II

- General method for constructing witnesses for states close to $|\Psi\rangle$

$$W = c \cdot 1 - \sum_k S_k$$

- Here S_k stabilize $|\Psi\rangle$

$$|\Psi\rangle = S_k |\Psi\rangle$$

Stabilizing operators $\left|\Psi\right\rangle = S_k \left|\Psi\right\rangle$

- For an N-qubit GHZ state

$$S_1 = \sigma_x^{(1)}\sigma_x^{(2)}\sigma_x^{(3)}\ldots\sigma_x^{(N)},$$
$$S_k = \sigma_z^{(k-1)}\sigma_z^{(k)}; \quad k = 2,3,\ldots,N.$$

For an N-qubit cluster state

$$S_1 = \sigma_x^{(1)}\sigma_z^{(2)},$$
$$S_k = \sigma_z^{(k-1)}\sigma_x^{(k)}\sigma_z^{(k+1)}; \quad k = 2,3,\ldots,N-1,$$
$$S_N = \sigma_z^{(N-1)}\sigma_x^{(N)}.$$

Cluster state

- Obtained from Ising spin chain dynamics
- For N=3 qubits it is equivalent to a GHZ state
- For N=4 qubits it is equivalent to

$$|C4\rangle = |0000\rangle + |1100\rangle + |0011\rangle - |1111\rangle$$

- See Briegel, Raussendorf, PRL 86, 910 (2001).

Stabilizer witnesses III

- Characteristics for our N-qubit entanglement witnesses
  - N locally measurable terms
  - Usually 2 (!!) measurement settings
  - For large N, tolerates noise $p_{noise} < 33\%$ (GHZ) / 25% (cluster)
  - For small N, noise tolerance is better (N=3; 40% / N=4; 33%)
  - Noise tolerance can be improved if more than N terms are included

Stabilizer witnesses IV

- Witness for N-qubit GHZ state

$$W_{GHZN} = 3 - 2\left[\frac{1 + S_1^{(GHZ_N)}}{2} + \prod_{k>1} \frac{1 + S_k^{(GHZ_N)}}{2}\right]$$

- Witness for N-qubit cluster state

$$W_{C_N} = 3 - 2\left[\prod_{k\ even} \frac{1 + S_k^{(C_N)}}{2} + \prod_{k\ odd} \frac{1 + S_k^{(C_N)}}{2}\right]$$

Stabilizer witnesses V

- Why do these witnesses detect genuine N-qubit entanglement? Because

$$W_{GHZN} - 2W_{GHZN}^{PROJ} \geq 0$$

$$\left(W_{GHZN}^{PROJ} = \frac{1}{2} \cdot 1 - \left| GHZ_N \right\rangle \left\langle GHZ_N \right| \right)$$

- Any state detected by our witness is also detected by the projector witness. Later detects genuine N-qubit entanglement.

Stabilizer witnesses VI

- The projector witness is also the sum of stabilizing operators

$$W_{GHZN}^{PROJ} = \frac{1}{2} \bullet 1 - \prod_k \frac{1 + S_k^{(GHZ_N)}}{2}$$

Number of measurement settings

$$W_{C_N} = 3 - 2\left[\prod_{k\ even} \frac{1+S_k^{(C_N)}}{2} + \prod_{k\ odd} \frac{1+S_k^{(C_N)}}{2}\right]$$

Z	X	Z	X	Z

X	Z	X	Z	X

$$W_{GHZN} = 3 - 2\left[\frac{1+S_1^{(GHZ_N)}}{2} + \prod_{k>1} \frac{1+S_k^{(GHZ_N)}}{2}\right]$$

X	X	X	X	X

Z	Z	Z	Z	Z

Noise

- In an experiment the GHZ state is never prepared perfectly

$$\rho = (1 - p_{noise}) |GHZ_3\rangle\langle GHZ_3| + p_{noise}\rho_{totally\_mixed}$$

- For each witness there is a noise limit. For a noise larger than this limit the GHZ state is not detected as entangled.

Noise tolerance

- Witness for N-qubit GHZ state
  - for N=3 : 40%
  - for large N : >33%

- Witness for N-qubit cluster state
  - for N=4 :33%
  - for large N : >25%

Connection to Bell inequalities

- Noise tolerance: 40% (2 settings)

$$W_{GHZ3} = \frac{3}{2} \bullet 1 - \sigma_x^1 \sigma_x^2 \sigma_x^3 - \frac{1}{2}\left[\sigma_z^1 \sigma_z^2 + \sigma_z^2 \sigma_z^3 + \sigma_z^1 \sigma_z^3\right]$$

- Noise tolerance: 50% (4 settings) Bell ineq.!

$$W'_{GHZ3} = 2 - \sigma_x^1 \sigma_x^2 \sigma_x^3 + \sigma_y^1 \sigma_y^2 \sigma_x^3 + \sigma_x^1 \sigma_y^2 \sigma_y^3 + \sigma_y^1 \sigma_x^2 \sigma_y^3$$

- Noise tolerance: 57% (4 settings) Projector!!

$$W_{GHZ3}^{PROJ} = W'_{GHZ3} + 1 - \sigma_z^1 \sigma_z^2 - \sigma_z^2 \sigma_z^3 - \sigma_z^1 \sigma_z^3$$

Some ideas for optical lattices

What can be measured in optical lattices of two-state atoms?

- The lattice of two-state atoms can be modelled as a spin chain. Only collective quantities can be measured:

$$J_{x/y/z} = \sum_k \sigma_{x/y/z}^{(k)}$$

Detecting the cluster state as entangled by collective measurement

- Two possibilities:

  (i) Measuring the expectation values, <Jx>, <Jy> and <Jz>, after some multi-qubit dynamics (like the previous example)

  (ii) Measuring <Jx>, <Jy> and <Jz> AND their moments

For the cluster state only (i) is possible.

(i) Measurement using multi-qubit dynamics

- For two stabilizing operators of the cluster state. For separable states

$$\langle S_k \rangle + \langle S_{k+1} \rangle = \left\langle \sigma_z^{(k-1)} \sigma_x^{(k)} \sigma_z^{(k+1)} \right\rangle + \left\langle \sigma_z^{(k)} \sigma_x^{(k+1)} \sigma_z^{(k+2)} \right\rangle$$

$$\leq \left\langle \sigma_x^{(k)} \right\rangle \left\langle \sigma_z^{(k+1)} \right\rangle + \left\langle \sigma_z^{(k)} \right\rangle \left\langle \sigma_x^{(k+1)} \right\rangle \leq 1$$

- For the sum of stabilizers: (N is even)

$$J := \left\langle \sum_k \sigma_z^{(k-1)} \sigma_x^{(k)} \sigma_z^{(k+1)} \right\rangle \leq \frac{N}{2}$$

How to measure with dynamics?

$$J := \left\langle \sum_k \sigma_z^{(k-1)} \sigma_x^{(k)} \sigma_z^{(k+1)} \right\rangle$$

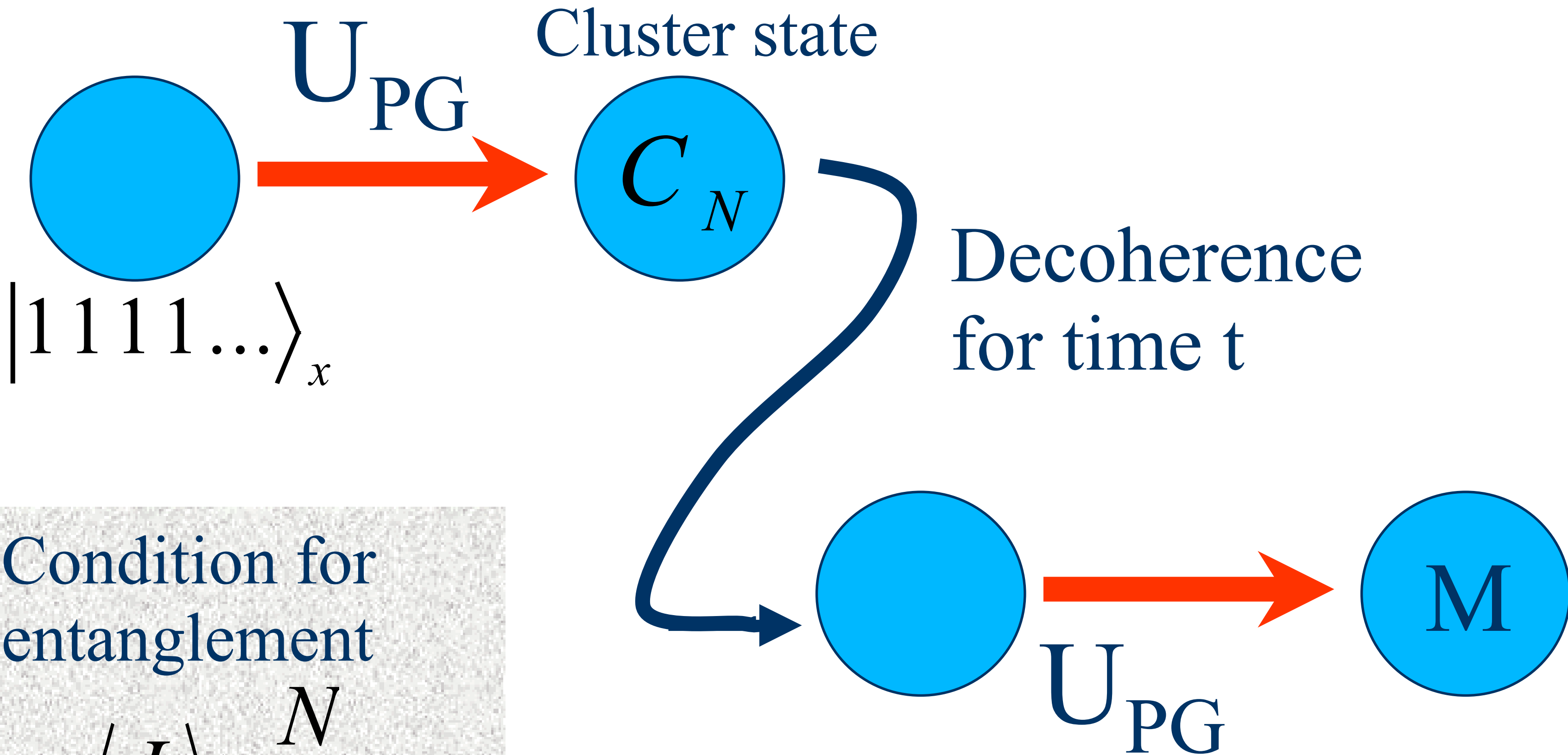
- We can measure J by measuring J_x after the dynamics U_{PG}.

$$\left\langle \sum_k \sigma_x^{(k)} \right\rangle_{after} = \left\langle U_{PG} \sum_k \sigma_x^{(k)} U_{PG} \right\rangle_{before} = \left\langle \sum_k \sigma_z^{(k-1)} \sigma_x^{(k)} \sigma_z^{(k+1)} \right\rangle_{before} =$$

- U_{PG} describes the application of the phase gate for all neighboring spins.

$$U_{PG} = e^{-i\frac{\pi}{4}\sum_k \left(\sigma_z^{(k)} - 1\right)\left(\sigma_z^{(k+1)} - 1\right)}$$

Poposed experiment for measuring the entanglement lifetime



Condition for entanglement

$$\langle J_x \rangle > \frac{N}{2}$$

Dynamics of $J := \left\langle \sum_k \sigma_z^{(k-1)} \sigma_x^{(k)} \sigma_z^{(k+1)} \right\rangle$ during decoherence

- Let us consider phase-flip channels acting in parallel

$$\varepsilon_k \rho = (1-p)\rho + p\sigma_z^k \rho \sigma_z^k \qquad p(t) = \frac{1-\exp(-\kappa t)}{2}$$

- J can be obtained with p as

$$J(p) = (1-2p)N$$

- The cluster state is detected as entangled if $p<0.25$.

(ii) Detecting entanglement without preceding dynamics

The moments of J_x, J_y and J_z for a (large enough) cluster state are the same as for the totally mixed state.

$$\left\langle J_x^2 \right\rangle_{cluster} = \frac{1}{4}\sum_{k,l}\left\langle \sigma_x^{(k)}\sigma_x^{(l)} \right\rangle_{cluster} = \frac{1}{4}\sum_{k,l}\left\langle \tilde{\sigma}_x^{(k)}\tilde{\sigma}_x^{(l)} \right\rangle_{|\uparrow\uparrow\uparrow\uparrow...\rangle}$$

$$\tilde{\sigma}_x^{(k)} = U_{PG}\sigma_x^{(k)}U_{PG} = \sigma_z^{(k-1)}\sigma_x^{(k)}\sigma_z^{(k+1)}$$

This term is 1 if $k=l$, otherwise it is 0 if $N>3$.

$$\left\langle J_x^2 \right\rangle_{cluster} = \frac{1}{4}\sum\left\langle \tilde{\sigma}_x^{(k)}\tilde{\sigma}_x^{(l)} \right\rangle_{|\uparrow\uparrow\uparrow...\rangle} = \frac{N}{4} + \frac{1}{4}\sum\left\langle \tilde{\sigma}_x^{(k)}\tilde{\sigma}_x^{(l)} \right\rangle_{|\uparrow\uparrow\uparrow...\rangle} = \frac{N}{4}$$

W state ($|W\rangle = |100\rangle + |010\rangle + |001\rangle$)

- The W state does not fit the stabilizer framework.Thus there are no locally measurable S_k's such that $|W\rangle = S_k|W\rangle$
- But the W state is uniquely defined by

$$\frac{1}{4}\left(\sigma_x^1\sigma_x^2 + \sigma_x^1\sigma_x^3 + \sigma_x^2\sigma_x^3 + \sigma_y^1\sigma_y^2 + \sigma_y^1\sigma_y^3 + \sigma_y^2\sigma_y^3\right)|W\rangle$$

$$= |W\rangle$$

$$\sigma_z^1\sigma_z^2\sigma_z^3|W\rangle = |W\rangle$$

Wstate II

- Ad-hoc witness (6 terms, 20% noise)

$$W_{W3} = (1+\sqrt{5}) - \sum_{k \neq l} \sigma_x^k \sigma_x^l - \sum_{k \neq l} \sigma_y^k \sigma_y^l$$

- Detects entangled states around

$$\left|W\right\rangle = \left|100\right\rangle + \left|010\right\rangle + \left|001\right\rangle$$

$$\left|\overline{W}\right\rangle = \left|011\right\rangle + \left|101\right\rangle + \left|110\right\rangle$$

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Summary

- Detection of genuine N-qubit entanglement was considered with few local measurements.
- The methods detect entangled states close to N-qubit GHZ and cluster states.
- Home page:
  http://www.mpq.mpg.de/Theorygroup/CIRAC/people/toth

- *************** THANK YOU!!! *************