Entanglement and permutational symmetry
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1 Motivation
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Symmetry is a central concept in quantum mechanics. Typically, the presence of some symmetry simplifies our calculations in physics.

A particular type of symmetry, permutational symmetry appears in many systems studied in quantum optics.

The separability problem is proven to be a very hard one. Thus, it is interesting to ask how permutational symmetry can simplify the separability problem.
Motivation

Entanglement criteria for bipartite systems

Symmetric bound entangled states–Bipartite case

Symmetric bound entangled states–Multipartite case
Consider two $d$-dimensional quantum systems. We will examine two types of permutational symmetries, denoting the corresponding sets by $\mathcal{I}$ and $\mathcal{S}$:

1. We call a state **permutationally invariant** (or just invariant, $\varrho \in \mathcal{I}$) if $\varrho$ is invariant under exchanging the particles. That is, $F\varrho F = \varrho$, where the flip operator is $F = \sum_{ij} |ij\rangle \langle ji|$. The reduced state of two randomly chosen particles of a larger ensemble has this symmetry.
Consider two $d$-dimensional quantum systems. We will examine two types of permutational symmetries, denoting the corresponding sets by $I$ and $S$:

1. We call a state **permutationally invariant** (or just invariant, $\varrho \in I$) if $\varrho$ is invariant under exchanging the particles. That is, $F\varrho F = \varrho$, where the flip operator is $F = \sum_{ij} |ij\rangle \langle ji|$. The reduced state of two randomly chosen particles of a larger ensemble has this symmetry.

2. We call a state **symmetric** ($\varrho \in S$) if it acts on the symmetric subspace only. This is the state space of two $d$-state bosons.

Clearly, we have $S \subset I$. 

Two types of symmetries
Expectation value matrix

Definition

The **expectation value matrix** of a bipartite quantum state is

\[ \eta_{kl}(\varrho) := \langle M_k \otimes M_l \rangle_\varrho, \]

where \( M_k \)'s are local orthogonal observables for both parties, satisfying

\[ \text{Tr}(M_k M_l) = \delta_{kl}. \]

We can decompose the density matrix as

\[ \varrho = \sum_{kl} \eta_{kl} M_k \otimes M_l. \]
Equivalence of several entanglement conditions for symmetric states

Observation 1. Let \( \rho \in S \) be a symmetric state. Then the following separability criteria are equivalent:

1. \( \rho \) has a positive partial transpose (PPT), \( \rho^{T_A} \geq 0 \).
2. \( \rho \) satisfies the Computable Cross Norm-Realignment (CCNR) criterion, \( \| R(\rho) \|_1 \leq 1 \), where \( R(\rho) \) is the realignment map and \( \|...\|_1 \) is the trace norm.
3. \( \eta \geq 0 \), or, equivalently \( \langle A \otimes A \rangle \geq 0 \) for all observables \( A \).
4. The correlation matrix, defined via the matrix elements as
   \[
   C_{kl} := \langle M_k \otimes M_l \rangle - \langle M_k \otimes 1 \rangle \langle 1 \otimes M_l \rangle
   \]
   is positive semidefinite: \( C \geq 0 \). [A.R. Usha Devi et al., Phys. Rev. Lett. 98, 060501 (2007).]
5. The state satisfies several variants of the Covariance Matrix Criterion (CMC). Latter are strictly stronger than the CCNR criterion for non-symmetric states.
Proof of Observation 1: Schmidt decomposition

Proof.

For invariant states, $\eta$ is a real symmetric matrix. It can be diagonalized by an orthogonal matrix $O$. The resulting diagonal matrix $\{\Lambda_k\}$ is the correlation matrix corresponding to the observables $M'_k = \sum O_{kl} M_l$. 

Hence, any invariant state can be written as $\rho = \sum k \Lambda_k M'_k \otimes M'_k$, where $M'_k$ are pairwise orthogonal observables. This is almost the Schmidt decomposition, however, $\Lambda_k$ can also be negative.

It can be shown that $-1 \leq \sum k \Lambda_k \leq 1$ for invariant states and $\sum k \Lambda_k = 1$ for symmetric states.
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It can be shown that $-1 \leq \sum_k \Lambda_k \leq 1$ for invariant states and $\sum_k \Lambda_k = 1$ for symmetric states.
The Computable Cross Norm-Realignment (CCNR) can be formulated as follows: If
\[ \sum_k |\Lambda_k| > 1 \]
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Proof of Observation 1: Equivalence of CCNR and $\eta \geq 0$

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  \[ \sum_k |\Lambda_k| > 1 \]
  in the Schmidt decomposition, then the quantum state is entangled.

- For symmetric states we have $\sum_k \Lambda_k = 1$, and $\sum_k |\Lambda_k| > 1$ is equivalent to
  \[ \Lambda_k < 0 \]
  for some $k$. Then $\langle M'_k \otimes M'_k \rangle < 0$ and $\eta$ has a negative eigenvalue.
Proof of Observation 1: CCNR–PPT equivalence

Let us take an alternative definition of the CCNR criterion.

- The CCNR criterion states that if $\mathcal{Q}$ is separable, then $\|R(\mathcal{Q})\|_1 \leq 1$ where the realigned density matrix is $R(\mathcal{Q}_{ij,kl}) = \mathcal{Q}_{ik,jl}$. This just means that if

$$\|(\mathcal{Q}F)^{T_A}\|_1 > 1$$

then $\mathcal{Q}$ is entangled.

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  then $\rho$ is entangled.


- Since for symmetric states

  $$\rho^F = \rho,$$

  this condition is equivalent to $\|\rho^{T_A}\|_1 > 1$. This is just the PPT criterion, since we have $\text{Tr}(\rho^{T_A}) = 1$. 

Proof of Observation 1: Equivalence of $C \geq 0$ and $\eta \geq 0$

- Now we show that $C \geq 0 \iff \eta \geq 0$.

- The direction $\Rightarrow$ is trivial, since for invariant states the matrix $\langle M_k \otimes 1 \rangle \langle 1 \otimes M_l \rangle$ is a projector and hence positive.

- The direction $\Leftarrow$: We make for a given state the special choice of observables $Q_k = M_k - \langle M_k \rangle$. Then, we just have $C(M_k) = \eta(Q_k)$.

- We know that $\eta(M_k) \geq 0 \Rightarrow \eta(Q_k) \geq 0$, even if $Q_k$ are not pairwise orthogonal observables. Hence $C(M_k) \geq 0$ follows.
Proof of Observation 1: Covariance Matrix Criterion

- Variants of the Covariance Matrix Criterion:

\[
\|C\|_1^2 \leq \left[ 1 - \text{Tr}(\varrho_A^2) \right] \left[ 1 - \text{Tr}(\varrho_B^2) \right]
\]

or

\[
2 \sum |C_{ii}| \leq \left[ 1 - \text{Tr}(\varrho_A^2) \right] + \left[ 1 - \text{Tr}(\varrho_B^2) \right].
\]

[O. Gühne et al., PRL 99, 130504 (2007); O. Gittsovich et al., PRA 78, 052319 (2008).]

- If \(\varrho\) is symmetric, the fact that \(C\) is positive semidefinite gives

\[
\|C\|_1 = \text{Tr}(C) = \sum \Lambda_k - \sum_k \text{Tr}(\varrho_A M'_k)^2 = 1 - \text{Tr}(\varrho_A^2) \quad \text{and similarly,}
\]

\[
\sum_i |C_{ii}| = \sum_i C_{ii} = 1 - \text{Tr}(\varrho_A^2).
\]

- Hence, a state fulfilling \(C \geq 0\) fulfills also both criteria. On the other hand, a state violating \(C \geq 0\) must also violate these criteria, as they are strictly stronger than the CCNR criterion.
Interesting result: For symmetric $\varrho$

$$\varrho^{T1} \geq 0 \iff \forall A : \langle A \otimes A \rangle \geq 0.$$  

This relates the positivity of partial transposition to the sign of certain two-body correlations.

Any symmetric state of the following form is PPT

$$\varrho_{\text{PPT}} = \sum_k p_k M_k \otimes M_k,$$  \hspace{1cm} (1)

where $p_k$ is a probability distribution, and $M_k$ are pairwise orthogonal observables, i.e., $\text{Tr}(M_k^2) = 1$. Compare this to the definition of separability

$$\varrho_{\text{sep}} = \sum_k p_k \varrho_k \otimes \varrho_k,$$  \hspace{1cm} (2)

where $\varrho_k$ are observables, $\text{Tr}(\varrho_k) = 1$, $\varrho_k \geq 0$ and $\varrho_k$ are pure, i.e, $\text{Tr}(\varrho_k^2) = 1$. 
Any symmetric state that can be written as

\[ \rho_{c+} = \sum_{k} c_k A_k \otimes A_k, \quad (3) \]

where \( c_k > 0 \), and \( A_k \) are some (not necessarily pairwise orthogonal) observables, is PPT. If \( \rho_{c+} \) is permutationally invariant, then it does not violate the CCNR criterion.

**Multipartite case:** A symmetric state of the form

\[ \rho_{\text{PPT}2:2} = \sum_{k} c_k A_k \otimes A_k \otimes A_k \otimes A_k \quad (4) \]

is PPT with respect to the 2:2 partition. Example: Smolin state.
Are there symmetric bound entangled states?

For symmetric states,

1. CCNR,
2. $\eta \geq 0$,
3. $C \geq 0$ and
4. CMC

are equivalent to the PPT criterion.

It is then quite hard to find symmetric PPT entangled states.

Do symmetric bound entangled states exist at all?
Symmetric bound entangled states–Bipartite case

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Symmetric bound entangled states

- Breuer presented, for even $d \geq 4$, a single parameter family of bound entangled states that are $I$ symmetric

  $$\rho_{B} = \lambda |\psi_{0}^{d}\rangle\langle\psi_{0}^{d}| + (1 - \lambda)\Pi_{S}^{d}.$$  


- The state is PPT entangled for $0 \leq \lambda \leq 1/(d + 2)$. Here $|\psi_{0}\rangle$ is the singlet state and $\Pi_{S}$ is the normalized projector to the symmetric subspace.

- Idea to construct bound entangled states with an $S$-symmetry: We embed a low dimensional entangled state into a higher dimensional Hilbert space, such that it becomes symmetric, while it remains entangled.
An $8 \times 8$ symmetric bound entangled states

We consider the symmetric state

$$\hat{\rho} = \lambda \Pi_{d_2}^a \otimes |\Psi_0^d\rangle\langle\Psi_0^d| + (1 - \lambda) \Pi_{d_2}^s \otimes \Pi_{d_2}^s.$$

Here, $\Pi_{d_2}^a$ and $\Pi_{d_2}^s$ are normalized projectors to the two-qudit symmetric/antisymmetric subspace with dimension $d_2$. Thus, $\hat{\rho}$ is symmetric.
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- If the original system is of dimension $d \times d$ then the system of $\hat{\varrho}$ is of dimension $dd_2 \times dd_2$. Since $\varrho_B$ is the reduced state of $\hat{\varrho}$, if the first is entangled, then the second is also entangled.

- For $d_2 = 2$ and $d = 4$, numerical calculation shows that $\hat{\varrho}$ is PPT for $\lambda < 0.062$.

This provides an example of an $S$ symmetric bound entangled state of size $8 \times 8$. 
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Symmetric bound entangled state via numerics–Basic idea

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Basic idea

- Let us consider an $N$-qubit symmetric state, that is, a state of the symmetric subspace. We consider even $N$.

- It is known that such a state is either separable with respect to all bipartitions or it is entangled with respect to all bipartitions.


- Thus any state that is PPT with respect to the $\frac{N}{2} : \frac{N}{2}$ partition while NPT with respect to some other partition is bound entangled with respect to the $\frac{N}{2} : \frac{N}{2}$ partition.
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- Since the state is symmetric, it can straightforwardly be mapped to a $(\frac{N}{2} + 1) \times (\frac{N}{2} + 1)$ bipartite symmetric state.
To obtain such a multiqubit state, one has to first generate an initial random state $\rho$ that is PPT with respect to the $\frac{N}{2} : \frac{N}{2}$ partition. Then, we compute the minimum nonzero eigenvalue of the partial transpose of $\rho$ with respect to all other partitions

$$\lambda_{\text{min}}(\rho) := \min_k \min_l \lambda_l(\rho^{T_k} I_k).$$

If $\lambda_{\text{min}}(\rho) < 0$ then the state is bound entangled with respect to the $\frac{N}{2} : \frac{N}{2}$ partition. If it is non-negative then we start an optimization process for decreasing this quantity. We generate another random density matrix $\Delta \rho$, and check the properties of $\rho' = (1 - \epsilon) \rho + \epsilon \Delta \rho$, (5) where $0 < \epsilon < 1$ is a small constant. If $\rho'$ is still PPT with respect to the $\frac{N}{2} : \frac{N}{2}$ partition and $\lambda_{\text{min}}(\rho') < \lambda_{\text{min}}(\rho)$ then we use $\rho'$ as our new random initial state $\rho$. 


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$$\rho' = (1 - \varepsilon)\rho + \varepsilon \Delta \rho,$$  \hspace{1cm} (5)

where $0 < \varepsilon < 1$ is a small constant. If $\rho'$ is still PPT with respect to the $\frac{N}{2} : \frac{N}{2}$ partition and $\lambda_{\text{min}}(\rho') < \lambda_{\text{min}}(\rho)$ then we use $\rho'$ as our new random initial state $\rho$. 


Repeating this procedure, we obtained a four-qubit symmetric state this way

\[ \rho_{BE4} = \begin{pmatrix}
0.22 & 0 & 0 & -0.059 & 0 \\
0 & 0.176 & 0 & 0 & 0 \\
0 & 0 & 0.167 & 0 & 0 \\
-0.059 & 0 & 0 & 0.254 & 0 \\
0 & 0 & 0 & 0 & 0.183
\end{pmatrix}. \]

The basis states are \(|0\rangle := |0000\rangle\), \(|1\rangle := \text{sym}(|1000\rangle)\), \(|2\rangle := \text{sym}(|1100\rangle)\), ...

The state is bound entangled with respect to the 2 : 2 partition. This corresponds to a 3 × 3 bipartite symmetric bound entangled state, demonstrating the simplest possible symmetric bound entangled state.
Our method can be straightforwardly generalized to create multipartite bound entangled states of the symmetric subspace, such that all bipartitions are PPT (“fully PPT states”).

We found such a state for five and six qubits.

Note that these states are both fully PPT and genuine multipartite entangled. It is further interesting to relate this to the Peres conjecture, stating that fully PPT states cannot violate a Bell inequality.
Conclusions

- In summary, we have discussed entanglement in symmetric systems.

- We showed that for states that are in the symmetric subspace several relevant entanglement conditions, especially the PPT criterion, the CCNR criterion, and the criterion based on covariance matrices matrices, coincide.

- We proved the existence of symmetric bound entangled states, in particular, a $3 \times 3$, five-qubit and six-qubit symmetric PPT entangled states.

- See G. Tóth and O. Gühne, PRL 102, 170503 (2009).

*** THANK YOU ***