Permutationally invariant quantum tomography

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Outline

1 Motivation
   - Why quantum tomography is important?

2 Quantum experiments with multi-qubit systems
   - Physical systems
   - Local measurements
   - Basic ideas and scaling

3 Permutationally invariant tomography
   - Main results
   - Example: XY PI tomography
   - Example: Experiment with a 4-qubit Dicke state

4 Extra slide 1: Number of settings
Why tomography is important?

- Many experiments aiming to create many-body entangled states.
- Quantum state tomography can be used to check how well the state has been prepared.
- However, the number of measurements scales exponentially with the number of qubits.
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Physical systems

State-of-the-art in experiments

- **14 qubits with trapped cold ions**

- **10 qubits with photons**

Full tomography:


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Only local measurements are possible

Definition

A single *local measurement setting* is the basic unit of experimental effort.

A local setting means measuring operator $A^{(k)}$ at qubit $k$ for all qubits.

\[
\begin{align*}
A^{(1)} & \quad A^{(2)} & \quad A^{(3)} & \quad \ldots & \quad A^{(N)} \\
\end{align*}
\]

- All two-qubit, three-qubit correlations, etc. can be obtained.

\[
\langle A^{(1)} A^{(2)} \rangle, \langle A^{(1)} A^{(3)} \rangle, \langle A^{(1)} A^{(2)} A^{(3)} \rangle \ldots
\]
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Approaches to solve the scalability problem

Problem: the number of settings needed for full tomography increases exponentially with the number of qubits.

Possible solutions:

- If the state is expected to be of a certain form (MPS), we can measure the parameters of the ansatz. 

- If the state is of low rank, we need fewer measurements. 

- We make tomography in a subspace of the density matrices (our approach).
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Permutationally invariant part of the density matrix:

\[ \rho_{\text{PI}} = \frac{1}{N!} \sum \Pi_k \rho \Pi_k^\dagger, \]

where \( \Pi_k \) are all the permutations of the qubits.

- Related literature: Reconstructing \( \rho_{\text{PI}} \) for spin systems.  

- Photons in a single mode optical fiber are always in a permutationally invariant state. Small set of measurements are needed for their characterization (experiments).  
Main results

Features of our method:

1. Is for spatially separated qubits.
2. Needs the minimal number of measurement settings.
3. Uses the measurements that lead to the smallest uncertainty possible of the elements of $\rho_{PI}$.
4. Gives an uncertainty for the recovered expectation values and density matrix elements.
5. If $\rho_{PI}$ is entangled, so is $\rho$. Can be used for entanglement detection!
Measurements

- We measure the same observable $A_j$ on all qubits. (Necessary for optimality.)

- Each qubit observable is defined by the measurement directions $\vec{a}_j$ using $A_j = a_{j,x}X + a_{j,y}Y + a_{j,z}Z$.

Number of measurement settings:

$$D_N = \binom{N+2}{N} = \frac{1}{2}(N^2 + 3N + 2).$$
What do we get from the measurements?

We obtain the expectation values for

$$\langle (A_j^{\otimes (N-n)} \otimes 1^{\otimes n})_{PI} \rangle$$

for $j = 1, 2, .., D_N$ and $n = 0, 1, ..., N$. 

How do we obtain the Bloch vector elements?

A Bloch vector element can be obtained as

$$\langle (X^{\otimes k} \otimes Y^{\otimes l} \otimes Z^{\otimes m} \otimes 1^{\otimes n})_{\text{PI}} \rangle = \sum_{j=1}^{D_N} c_j^{(k,l,m)} \times \langle (A_j^{\otimes (N-n)} \otimes 1^{\otimes n})_{\text{PI}} \rangle.$$ 

- Coefficients are not unique if $n > 0$. 
The uncertainty of the reconstructed Bloch vector element is

$$\mathcal{E}^2[(X^{\otimes k} \otimes Y^{\otimes l} \otimes Z^{\otimes m} \otimes 1^{\otimes n})_{\text{PI}}] = \sum_{j=1}^{D_N} |c^{(k,l,m)}_j| \mathcal{E}^2[(A_j^{\otimes (N-n)} \otimes 1^{\otimes n})_{\text{PI}}].$$

For a fixed set of $A_j$, we have a formula to find the $c^{(k,l,m)}_j$’s giving the minimal uncertainty.
Optimization for $A_j$

We have to find $D_N$ measurement directions $\vec{a}_j$ on the Bloch sphere minimizing the variance

$$(\mathcal{E}_{\text{total}})^2 = \sum_{k+l+m+n=N} \mathcal{E}^2 \left[ (X^\otimes k \otimes Y^\otimes l \otimes Z^\otimes m \otimes 1^\otimes n)_{\Pi} \right] \times \left( \frac{N!}{k!l!m!n!} \right).$$
Summary of algorithm

### Obtaining the "total uncertainty" for given measurements

\[ \rho_0, \quad \text{the state we expect} \]
\[ A_j, \quad \text{what we measure} \]

\[ \Rightarrow \quad \text{BOX #1} \quad \Rightarrow \quad (\epsilon_{\text{total}})^2 \]

### Evaluating the experimental results

\[ \text{measurement results} \]
\[ A_j \]

\[ \Rightarrow \quad \text{BOX #2} \quad \Rightarrow \quad \left\{ \begin{array}{l} \text{Bloch vector elements} \\ \text{variances} \end{array} \right\} \]
How much is the information loss?

Estimation of the fidelity $F(\varrho, \varrho_{PI})$:

$$F(\varrho, \varrho_{PI}) \geq \langle P_s \rangle_\varrho^2 = \langle P_s \rangle_{\varrho_{PI}}^2,$$

where $P_s$ is the projector to the $N$-qubit symmetric subspace.

- $F(\varrho, \varrho_{PI})$ can be estimated only from $\varrho_{PI}$!

Proof: using the theory of angular momentum for qubits.

Similar formalism appear concerning handling multi-copy qubit states:


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Let us assume that we want to know only the expectation values of operators of the form

$$\langle A(\phi)^{\otimes N} \rangle$$

where

$$A(\phi) = \cos(\phi)\sigma_x + \sin(\phi)\sigma_y.$$  

The space of such operators has dimension $N + 1$. We have to choose $\{\phi_j\}_{j=1}^{N+1}$ angles for the $\{A_j\}_{j=1}^{N+1}$ operators we have to measure.
Let us assume that we measure
\[ \langle A_j^{\otimes N} \rangle \]
for \( j = 1, 2, ..., N + 1 \).

Reconstructed values and uncertainties
\[ \langle A(\phi) \otimes^N \rangle = \sum_{j=1}^{N+1} c_j^{(\phi)} \times \langle A_j^{\otimes N} \rangle. \]

Reconstructed coefficients \( c_j^{(\phi)} \) \times \text{Measured data} \]

Let us assume that all of these measurements have a variance \( \Delta^2 \).
Simple example: XY PI tomography III

- Numerical example for \( N = 6 \).

Random directions \( \phi_j \)  
Uncertainty of \( A(\phi)^\otimes N \)  
Uniform directions
Numerical example for $N = 6$. This random choice is even worse ...

Random directions $\phi_j$  
Uncertainty of $A(\phi)^{\otimes N}$  
Uniform directions
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4-qubit Dicke state, optimized settings (exp.)

*** NEXT TALK by Christian Schwemmer ***
We determined the optimal $A_j$ for p.i. tomography for $N = 4, 6, ..., 14$. The maximal squared uncertainty of the Bloch vector elements is

$$\epsilon_{\text{max}}^2 = \max_{k,l,m,n} \mathcal{E}^2[\left( X^{\otimes k} \otimes Y^{\otimes l} \otimes Z^{\otimes m} \otimes \mathbb{I}^{\otimes n} \right)_{\text{PI}}]$$

(Total count is the same as in the experiment: 2050.)
Operator expectation values can be recovered directly from the measurement data as

$$\langle Op \rangle = \sum_{j=1}^{D_N} \sum_{n=1}^{N} c_{j,n}^{Op} \langle (A_j^{\otimes (N-n)} \otimes \mathbb{I}^\otimes n)_{PI} \rangle,$$

where the $c_{j,n}^{Op}$ are constants.

$Op = |D_N^{(N/2)} \rangle \langle D_N^{(N/2)}|$, blue: $\varrho_0 \propto \mathbb{I}$, red: upper bound for any $\varrho_0$. 

![Graph showing expected results with N on the x-axis and ξ on the y-axis, with blue and red markers representing different conditions.](image-url)
Comparison with other methods for efficient tomography

- If a state is detected as entangled, it is surely entangled. No assumption is used concerning the form of the quantum state.

- Expectation values of all permutationally invariant operators are the same for $\rho$ and $\rho_{PI}$.

- Typically, it can be used in experiments in which permutationally invariant states are created.
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Summary

- We discussed permutationally invariant tomography for large multi-qubits systems.
- It paves the way for quantum experiments with more than 6 – 8 qubits.

See:

THANK YOU FOR YOUR ATTENTION!
How many settings we need?

- Expectation values of \((X^\otimes k \otimes Y^\otimes l \otimes Z^\otimes m \otimes 1^\otimes n)_{PI}\) are needed.

- For a given \(n\), the dimension of this subspace is \(D_{(N-n)}\) (simple counting).

- Operators with different \(n\) are orthogonal to each other.

- Every measurement setting gives a single real degree of freedom for each subspace.

- Hence the number of settings cannot be smaller than the largest dimension, which is \(D_N\).