Extremal properties of the variance and the quantum Fisher information

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DPG Meeting, Hannover, 18 March 2013
1 Motivation
   - Why variance and the quantum Fisher information are important?

2 Variance and quantum Fisher information
   - Basic definitions
   - Entanglement detection with the variance
   - Entanglement detection with the quantum Fisher information

3 Generalized variance and quantum Fisher information
   - Generalized variance
   - Generalized quantum Fisher information
   - Generalized quantities in the literature
Why variance and the quantum Fisher information is important?

- Variance appears in all areas of physics.
- Quantum Fisher information is a central notion in metrology.
- Concave roofs, convex roofs are interesting in entanglement theory.
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The variance is defined as

$$(\Delta A)^2 = \langle A^2 \rangle - \langle A \rangle^2.$$ 

The variance is concave.
Quantum Fisher information (QFI)

- The parameter $\theta$ must be estimated by measuring the output state:

\[ \rho_{\text{output}} = U(\theta) \rho U(\theta)^\dagger = \exp(-iA\theta) \rho \exp(iA\theta) \]

- Cramér-Rao bound

\[ \Delta \theta \geq \frac{1}{\sqrt{F_Q^{\text{usual}}[\rho, A]}}. \]

- The quantum Fisher information is

\[ F_Q^{\text{usual}}[\rho, A] = 2 \sum_{i,j} \frac{(\lambda_i - \lambda_j)^2}{\lambda_i + \lambda_j} |A_{ij}|^2. \]

- For pure states, $F_Q^{\text{usual}}[\rho, A] = 4(\Delta A)^2$, and it is convex.

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Two properties of the variance are used:

- For pure states, it is $\langle A^2 \rangle_\psi - \langle A \rangle_\psi^2$.
- It is concave.

Any other quantity with these properties could be used instead of the variance.

If it were smaller than the variance, then it would even be better than the variance for this purpose.

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Entanglement detection with the QFI

Two properties of the QFI are used:

- For pure states, it is $4(\langle A^2 \rangle_\psi - \langle A \rangle^2_\psi)$.
- It is convex.

Any other quantity with these properties could be used instead of the QFI.

If it were larger than the usual quantum Fisher information, then it would even be better for this purpose.

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Generalized variance

**Definition 1.** Generalized variance \( \text{var}_\varrho(A) \) is defined as follows.

1. For pure states, we have
   \[
   \text{var}_\psi(A) = (\Delta A)^2_\psi.
   \]

2. For mixed states, \( \text{var}_\varrho(A) \) is concave in the state.

**Definition 2.** The minimal generalized variance \( \text{var}^\text{min}_\varrho(A) \) is defined as follows.

1. For pure states, it equals the usual variance
   \[
   \text{var}^\text{min}_\psi(A) = (\Delta A)^2_\psi,
   \]

2. For mixed states, it is defined through a concave roof construction
   \[
   \text{var}^\text{min}_\varrho(A) = \sup_{\{p_k, \psi_k\}} \sum_k p_k (\Delta A)^2_{\psi_k},
   \]
   where
   \[
   \varrho = \sum_k p_k |\psi_k\rangle \langle \psi_k|.
   \]
Theorem 1

The minimal generalized variance is the usual variance

\[ \text{var}_\varphi^\text{min}(A) = (\Delta A)^2. \]

In other words, the variance its own concave roof.

**Hand waving proof:**

\[ (\Delta A)^2 = \sum_k p_k (\Delta A)^2 \psi_k + (\langle A \rangle \psi_k - \langle A \rangle \varphi)^2. \]

You can always find a decomposition such that \( \langle A \rangle \psi_k = \langle A \rangle \varphi \) for all \( k \).
Theorem 1

Hand waving proof, continuation; geometric argument:

For details, please see arxiv:1109.2831.
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Generalized quantum Fisher information

**Definition 3.** Generalized quantum Fisher information $F_Q[\rho, A]$:  
1. For pure states, we have 
   \[ F_Q[\rho, A] = 4(\Delta A)^2_\psi. \]
   The factor 4 appears for historical reasons.
2. For mixed states, $F_Q[\rho, A]$ is convex in the state.

**Definition 4.** Maximal quantum Fisher information $F_Q^{\text{max}}[\rho, A]$:  
1. For pure states, it equals four times the usual variance 
   \[ F_Q^{\text{max}}[\rho, A] = 4(\Delta A)^2_\psi. \]
2. For mixed states, it is defined through a convex roof construction 
   \[ F_Q^{\text{max}}[\rho, A] = 4 \inf_{\{p_k, \psi_k\}} p_k(\Delta A)^2_\psi_k. \]
Theorem 2

For rank-2 states

\[ F_Q^{\text{max}} [\rho, A] = F_Q^{\text{usual}} [\rho, A]. \]

For an analytic proof, see G. Tóth and D. Petz, arxiv:1109.2831.

In other words, the quantum Fisher information is four times the convex roof of the variance for rank-2 states.
Numerics for rank $>2$

- The maximal generalized q. Fisher information can be written as

$$F_Q^\text{max} [\varrho, A] = 4 \left( \langle A^2 \rangle_{\varrho} - \sup_{\{p_k, |\psi_k\rangle\}} \sum_k p_k \langle A \rangle_{\psi_k}^2 \right).$$

- Rewriting the term quadratic in expectation values as an operator acting on a bipartite system

$$F_Q^\text{max} [\varrho, A] = 4 \left( \langle A^2 \rangle_{\varrho} - \sup_{\{p_k, |\psi_k\rangle\}} \sum_k p_k \langle A \otimes A \rangle_{\psi_k \otimes \psi_k} \right).$$

- Further transformations lead to

$$F_Q^\text{max} [\varrho, A] = 4 \left( \langle A^2 \rangle_{\varrho} - \sup_{\{p_k, |\psi_k\rangle\}} \sum_k p_k \langle A \otimes A \rangle \sum_k p_k |\psi_k\rangle \langle \psi_k| \otimes |\psi_k\rangle \langle \psi_k| \right).$$
Hence we obtain that

$$F^\text{max}_Q[\rho, A] = 4\left(\langle A^2 \rangle_{\rho} - \sup_{\rho_{ss} \in S_s, \quad Tr_1(\rho_{ss})=\rho} \langle A \otimes A \rangle_{\rho_{ss}}\right),$$

where $S_s$ are symmetric separable states.

Instead of the separable states, we can do the optimization for PPT states or states with a PPT symmetric extension.

Extensive numerics on random $\rho$ and $A$ confirm that

$$F_Q = F^\text{max}_Q$$

holds within a large degree of accuracy.
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Generalized variance and quantum Fisher information in the literature


- Defines a variance and a corresponding quantum Fisher information for each standard matrix monotone function $f : \mathbb{R}^+ \rightarrow \mathbb{R}^+$.

- **Surprisingly**, his variances and quantum Fisher information definitions fit the definitions of this presentation.

- Our quantities are extremal even within the sets defined by Petz et al. However, our definitions are broader.

**Conjecture**

We conjecture that

\[ F_Q = F_Q^{\text{max}} \]

for density matrices of any rank and for any Hermitian \( A \).

Conjecture based on

- Analytics for rank 2
- Extensive numerics for rank>2
- Statement is true for a large subset

**Follow-up**

- Proof: Sixia Yu, arxiv 1302.5311.
- We should look for connections to

Summary

- We defined the generalized variance and the generalized quantum Fisher information.
- We found that the variance is its own concave roof, while the quantum Fisher information is its own convex roof.

See:
G. Tóth and D. Petz,
Extremal properties of the variance and the quantum Fisher information, Phys. Rev. A, in press;
arxiv:1109.2831.

THANK YOU FOR YOUR ATTENTION!