Multipartite entanglement and high precision metrology

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Outline

1 Motivation
- Why the connection between multipartite entanglement and Fisher information is important?

2 Metrology and multipartite entanglement
- Quantum Fisher information
- Properties of the Quantum Fisher information
- Quantum Fisher information and entanglement
Why the connection between multipartite entanglement and Fisher information is important?

- Genuine multipartite entanglement appears often in quantum information.

- While bipartite entanglement is quite well understood, the role of multipartite entanglement is not so clear.

- Thus, it is very interesting if we can show that it has a central role in metrology.
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Metrology and multipartite entanglement in the literature


- Multipartite entanglement, not simple nonseparability, is needed for extreme spin squeezing, which can be applied in spectroscopy and atomic clocks. A.S. Sørensen and K. Mølmer, Phys. Rev. Lett. 86, 4431 (2001).

- Not all entangled states are useful for phase estimation, at least in a linear interferometer. P. Hyllus, O. Gühne, and A. Smerzi, 82, 012337 (2009).
Let us consider the following process:

\[ U = \exp(-iJ_n \Theta) \]

The dynamics described above is \( \rho_{\text{out}} = e^{-i\theta J_n} \rho e^{i\theta J_n} \).

We would like to determine the angle \( \theta \) by measuring \( \rho_{\text{out}} \).
The phase estimation sensitivity is limited as

\[ \Delta \theta \geq \frac{1}{\sqrt{F_Q[\rho, J_{\vec{n}}]}} \]

where \( F_Q \) is the quantum Fisher information, \( \rho \) is a quantum state and \( J_{\vec{n}} \) is a collective angular momentum component.

The Braunstein-Caves quantum Fisher information is

\[ F[\rho, X] = \sum_{ij} \frac{2(\lambda_i - \lambda)^2}{\lambda_i + \lambda_j} |X_{ij}|^2. \]

C.W. Helstrom, *Quantum Detection and Estimation Theory* (1976),
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Two important properties:

1. For a pure state $\varrho$, we have $F[\varrho, J_l] = 4(\Delta J_l)_\varrho^2$.

2. $F[\varrho, J_l]$ is convex in the state, that is
   $$F[p_1\varrho_1 + p_2\varrho_2, J_l] \leq p_1 F[\varrho_1, J_l] + p_2 F[\varrho_2, J_l].$$

It also follows that $F[\varrho, J_l] \leq 4(\Delta J_l)_\varrho^2$.

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For $N$-qubit separable states we have

$$F_Q[\varrho, J_I] \leq N.$$  

Here, $J_I = \frac{1}{2} \sum_{k=1}^{N} \sigma_l^{(k)}$ where $\sigma_l^{(k)}$ are the Pauli spin matrices. The maximum for the left-hand side is $N^2$.

Thus, for separable states

$$\Delta \theta \geq \frac{1}{\sqrt{N}},$$

while for entangled states

$$\Delta \theta \geq \frac{1}{N}.$$
Observation 1

For N-qubit separable states we have

\[ \sum_{l=x, y, z} F_Q[\varrho, J_l] \leq 2N. \]  

(1)

- Eq. (1) is a condition for the average sensitivity of the interferometer. All states violating Eq. (1) are entangled.

GT, PRA 85, 022322 (2012); P. Hyllus et al., PRA 85, 022321 (2012).
Observation 2

For quantum states we have the bound

\[
\sum_{l=x,y,z} F_{Q[\rho, J_l]} \leq N(N + 2).
\] (2)

GHZ states and \(N\)-qubit symmetric Dicke states with \(\frac{N}{2}\) excitations saturate Eq. (2).

- Dicke states have been investigated recently in several experiments.

- In general, pure symmetric states for which \(\langle J_l \rangle = 0\) for \(l = x, y, z\) saturate Eq. (2).

GT, PRA 85, 022322 (2012); P. Hyllus et al., PRA 85, 022321 (2012).
Quantum Fisher information and multipartite entanglement

Next, we will consider $k$-producible or $k$-entangled states:

**Observation 3**

For $N$-qubit $k$-producible states

$$
\sum_{l=x,y,z} F_Q[\rho, J_l] \leq nk(k + 2) + (N - nk)(N - nk + 2).
$$

where $n$ is the integer part of $\frac{N}{k}$. For the $k = N - 1$ case, this bound can be improved

$$
\sum_{l=x,y,z} F_Q[\rho, J_l] \leq N^2 + 1. \tag{3}
$$

Eq. (3) is also the inequality for biseparable states. Any state that violates Eq. (3) is genuine multipartite entangled.
Genuine multipartite entanglement, not simple nonseparability is needed to achieve maximum sensitivity in a linear interferometer.
Figure: Points in the \( (F_Q[\rho, J_x], F_Q[\rho, J_y], F_Q[\rho, J_z]) \)-space for \( N = 6 \).

- Points corresponding to separable states are not above the \( S_x - S_y - S_z \) plane.
- Points corresponding to biseparable states are not above the \( G_x - G_y - G_z \) plane.
A completely mixed state

\[ \rho_C = \frac{1}{2^N}. \]

corresponds to the point \( C(0, 0, 0) \).

States corresponding to the point \( S_x(0, N, N) \) is

\[ |\psi\rangle_{S_i} = |+\rangle_x^{\otimes N/2} \otimes | -\rangle_x^{\otimes N/2}. \]

\( S_y \) and \( S_z \) are similar.
Which part of the space corresponds to quantum states? - Points II

- $D_z$ : $N$-qubit symmetric Dicke state with $\frac{N}{2}$ excitations.

$$|D_{N}^{(N/2)}\rangle = \left(\frac{N}{N/2}\right)^{-\frac{1}{2}} \sum_{k} \mathcal{P}_{k}\{|0\rangle^{\otimes \frac{N}{2}} \otimes |1\rangle^{\otimes \frac{N}{2}}\},$$

where $\sum_{k} \mathcal{P}_{k}$ denotes summation over all possible permutations.

- $N$-qubit GHZ states

$$|\psi\rangle_{GHZ_{z}} = \frac{1}{\sqrt{2}} \left(|0\rangle^{\otimes N} + |1\rangle^{\otimes N}\right).$$
Which part of the space corresponds to quantum states? - 2D polytopes

- For all points in the $S_x, S_y, S_z$ polytope, there is a corresponding pure product state for even $N$.

- For given $F[\varrho, J_l]$ for $l = x, y, z$, such a state is defined as

\[
\varrho = \left[ \frac{1}{2} + \frac{1}{2} \sum_{l=x,y,z} c_l \sigma_l \right] \otimes \left[ \frac{1}{2} - \frac{1}{2} \sum_{l=x,y,z} c_l \sigma_l \right] \otimes \frac{N}{2},
\]

where $c_l^2 = 1 - \frac{F_0[\varrho, J_l]}{N}$, where $\sum_l c_l^2 = 1$. 

Which part of the space corresponds to quantum states? - 2D polytopes II

Figure: Randomly chosen points in the \((F_Q[\rho, J_x], F_Q[\rho, J_y], F_Q[\rho, J_z])\)-space corresponding to states \(|\Psi(\alpha_x, \alpha_y, \alpha_z)\rangle\) for \(N = 8\).

- All the points are in the plane of \(D_x, D_y\) and \(D_z\).
Which part of the space corresponds to quantum states? - 3D polytopes

- A pure state mixed with the completely mixed state

\[ \rho^{\text{mixed}}(p) = p\rho + (1 - p)\frac{1}{2^N} \]

- The states \( \rho^{\text{mixed}}(p) \) are on a straight line on our figures.
Observation 5. If $N$ is even, then there is a separable state for each point in the $S_x, S_y, S_z, C$ polytope.
Observation 6. If $N$ is divisible by 4, then for all the points of the $D_x, D_y, D_z, G_x, G_y, G_z$ polytope, there is a quantum state with genuine multipartite entanglement.
Summary

- We defined entanglement conditions in terms of the quantum Fisher information.

- We showed that genuine multipartite entanglement is needed for maximum metrological sensitivity.

See:
G. Tóth, PRA 85, 022322 (2012).


THANK YOU FOR YOUR ATTENTION!