Optimal generalized variance and quantum Fisher information

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Outline

1 Motivation
   - Why variance and the quantum Fisher information is important?

2 Variance and quantum Fisher information
   - Basic definitions
   - Entanglement detection with the variance
   - Entanglement detection with the quantum Fisher information

3 Generalized variance and quantum Fisher information
Why variance and the quantum Fisher information is important?

- Variance is a quantity appearing often in all areas of physics.
- Quantum Fisher information is an important notion in metrology. Any connection between the two is interesting.
- Concave roofs, convex roofs are also interesting - they are typically difficult to compute.
Motivation
- Why variance and the quantum Fisher information is important?

Variance and quantum Fisher information
- Basic definitions
- Entanglement detection with the variance
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Generalized variance and quantum Fisher information
The variance is defined as

\[ (\Delta A)^2 = \langle A^2 \rangle - \langle A \rangle^2. \]

The variance is \textit{concave}.
Quantum Fisher information

- The small parameter $\theta$ must be estimated by making measurements on the output state:

$$Q \xrightarrow{U(\theta) = \exp(-iA\theta)} Q_{\text{output}}$$

- Cramér-Rao bound

$$\Delta \theta \geq \frac{1}{\sqrt{F_Q^{\text{usual}}[Q, A]}}.$$

- The quantum Fisher information is

$$F_Q^{\text{usual}}[Q, A] = 2 \sum_{i,j} \frac{(\lambda_i - \lambda_j)^2}{\lambda_i + \lambda_j} |A_{ij}|^2.$$

- For pure states, $F_Q^{\text{usual}}[Q, A] = 4(\Delta A)^2_Q$, and it is convex.
Motivation
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Generalized variance and quantum Fisher information
Entanglement detection with the variance

- For a product states \( \varrho = \varrho_1 \otimes \varrho_2 \) we have

\[
(\Delta(A_1 \otimes 1 + 1 \otimes A_2))^2_{\varrho} = (\Delta A_1)^2_{\varrho_1} + (\Delta A_2)^2_{\varrho_2}.
\]

Here, \( A_1 \) and \( A_2 \) act on the first and second subsystem, respectively.

- \( B_1 \) and \( B_2 \), acting on the same subsystems. They fulfill the uncertainty relations

\[
(\Delta A_k)^2_{\varrho_k} + (\Delta B_k)^2_{\varrho_k} \geq L_k,
\]

where \( L_k \) are some constants.

- Hence, for product states

\[
(\Delta(A_1 \otimes 1 + 1 \otimes A_2))^2_{\varrho} + (\Delta(B_1 \otimes 1 + 1 \otimes B_2))^2_{\varrho} \geq L_1 + L_2.
\]

- Due to convexity, also true for separable states.

Note that only two properties of the variance were used:

- For pure states, it is \( \langle A^2 \rangle_\psi - \langle A \rangle_\psi^2 \).
- It is concave.

Another quantity with these properties could also be used for entanglement detection.

If it were smaller than the variance, then it would even be better than the variance for this purpose.
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3 Generalized variance and quantum Fisher information
Entanglement detection with the q. Fisher information

- For our bipartite system, for pure states we have

\[(\Delta A_k)^2_{\rho_k} \leq M_k,\]

where \(M_k\) are some constants.

- Based on these, for pure product state we have

\[(\Delta (A_1 \otimes 1 + 1 \otimes A_2))^2_{\rho} \leq M_1 + M_2,\]

- Then, due to the convexity of the quantum Fisher information, for mixed separable states we have.

\[F_{Q}^{\text{usual}}[\rho, A_1 \otimes 1 + 1 \otimes A_2] \leq 4(M_1 + M_2).\]

Any state that violates this is entangled.

Note that only two properties of the quantum Fisher information were used:

- For pure states, it is $\langle A^2 \rangle_\psi - \langle A \rangle^2_\psi$.
- It is convex.

Another quantity with these properties could also be used for entanglement detection.

If it were larger than the usual quantum Fisher information, then it would even be better for this purpose.
**Generalized variance**

**Definition 1.** Generalized variance $\text{var}_\varrho(A)$ is defined as follows.

1. For pure states, we have
   \[
   \text{var}_\psi(A) = (\Delta A)^2_\psi.
   \]

2. For mixed states, $\text{var}_\varrho(A)$ is concave in the state.

**Definition 2.** The minimal generalized variance $\text{var}_\varrho^{\text{min}}(A)$ is defined as follows.

1. For pure states, it equals the usual variance
   \[
   \text{var}_\psi^{\text{min}}(A) = (\Delta A)^2_\psi,
   \]

2. For mixed states, it is defined through a concave roof construction
   \[
   \text{var}_\varrho^{\text{min}}(A) = \sup_{\{p_k, \psi_k\}} \sum_k p_k (\Delta A)^2_{\psi_k},
   \]
   where
   \[
   \varrho = \sum_k p_k |\psi_k\rangle \langle \psi_k|.
   \]
**Theorem 1.** The minimal generalized variance is the usual variance

\[ \text{var}_{\varrho}^\text{min}(A) = (\Delta A)^2 \varrho. \]

In other words, the variance is its own concave roof.

**Handwaving proof:**

\[ (\Delta A)_\varrho^2 = \sum_k p_k (\Delta A)^2 \psi_k + (\langle A \rangle \psi_k - \langle A \rangle_\varrho)^2. \]

You can always find a decomposition such that \( \langle A \rangle \psi_k = \langle A \rangle_\varrho \) for all \( k \).
Theorem 1

Handwaving proof, continuation, Geometric argument:

\[ \text{Set of quantum states} \]

\[ \rho_0 \]

\[ \text{Tr}(A\rho) = A_0 \]

For details, please see arxiv:1109.2831.
Generalized quantum Fisher information

**Definition 3.** The generalized quantum Fisher information $F_Q[\rho, A]$ is defined as follows.

1. For pure states, we have

   $$F_Q[\rho, A] = 4(\Delta A)^2_\psi.$$

   The factor 4 appears for historical reasons.

2. For mixed states, $F_Q[\rho, A]$ is convex in the state.

**Definition 4.** $F_Q^{\text{max}}[\rho, A]$ is defined as follows.

1. For pure states, it equals four times the usual variance

   $$F_Q^{\text{max}}[\rho, A] = 4(\Delta A)^2_\psi.$$

2. For mixed states, it is defined through a convex roof construction

   $$F_Q^{\text{max}}[\rho, A] = 4 \inf_{\{p_k, \psi_k\}} p_k(\Delta A)^2_{\psi_k}.$$
Theorem 2. The maximal generalized quantum Fisher information is the usual quantum Fisher information for rank-2 states.

\[ F_{Q}^{\text{max}}[\rho, A] = F_{Q}^{\text{usual}}[\rho, A] \]

In other words, the quantum Fisher information is the convex roof of the variance for rank-2 states.

It would be interesting to find connection to the statements of [B.M. Escher, R.L. de Matos Filho, and L. Davidovich, Nature Phys. (2011)] concerning quantum Fisher information and purifications.
(For the idea, thanks to Rafal Demkowicz-Dobrzanski.)
Generalized variance and quantum Fisher information in the literature

- D. Petz defined before generalized variances and quantum Fisher informations.

- He presents formulas, that define a variance and a corresponding quantum Fisher information for each standard matrix monotone function $f : \mathbb{R}^+ \to \mathbb{R}^+$.

- Surprisingly, his variances and quantum Fisher informations definitions fit the definitions of this presentation.

Summary

- We discussed how to define the generalized variance and the generalized quantum Fisher information.
- We found that the variance is its own concave roof, while the quantum Fisher information is its own convex roof for rank-2 states.

See:
G. Tóth and D. Petz,
Optimal generalized variance and quantum Fisher information,
arxiv:1109.2831.

THANK YOU FOR YOUR ATTENTION!