Witnessing Entanglement with the Stabilizer Formalism

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Outline

- Posing the problem.
- Stabilizing operators for GHZ and cluster states
- Entanglement conditions with stabilizing operators
- Characterizing entanglement conditions: Noise tolerance
- Criteria for detecting genuine multi-qubit entanglement (or estimating the fidelity)
- Nonlinear entanglement criteria

Main problems in a *many*-qubit system

- Let us consider systems in which the qubits are individually accessible, but only *local* measurements are possible.
- Complete state tomography is very hard since the number of measurements increases *exponentially* with the number of qubits.
- Measuring a usual entanglement witness or the fidelity with respect to a given state is also very hard.
- We look for a solution using stabilizer theory. This makes it possible to characterize the state with few measurements.

Stabilizing operators

- An operator S_k *stabilizes* the quantum state $|\Psi\rangle$, if

$$|\Psi\rangle = S_k |\Psi\rangle$$

- Many quantum states can be more efficiently described by their stabilizing operators, than by the state vector.
- This is used in error correction [Gottesman PRA 96].

Stabilizing operators for the GHZ state

- Example: GHZ state: $|000\rangle + |111\rangle$

Stabilized by:
$$S_1 = X^{(1)} X^{(2)} X^{(3)},$$
$$S_2 = Z^{(1)} Z^{(2)},$$
$$S_3 = Z^{(2)} Z^{(3)}.$$

- The GHZ state is *uniquely* characterized by

$$\langle S_1 \rangle = \langle S_2 \rangle = \langle S_3 \rangle = +1.$$

Our first sufficient entanglement condition

- For pure product states

$$\langle S_1 + S_2 \rangle = \langle X^{(1)} X^{(2)} X^{(3)} + Z^{(1)} Z^{(2)} \rangle$$
$$= \langle X^{(1)} \rangle \langle X^{(2)} \rangle \langle X^{(3)} \rangle + \langle Z^{(1)} \rangle \langle Z^{(2)} \rangle$$
$$\leq \left| \langle X^{(1)} \rangle \langle X^{(2)} \rangle \right| + \left| \langle Z^{(1)} \rangle \langle Z^{(2)} \rangle \right| \leq 1$$

- Due to the convexity of separable states, this is also true for mixed separable states.

Our first sufficient entanglement condition II

- For separable states

$$\left\langle X^{(1)} X^{(2)} X^{(3)} + Z^{(1)} Z^{(2)} \right\rangle \leq 1$$

If this bound is violated then the state is entangled.

- For the GHZ state

$$\left\langle X^{(1)} X^{(2)} X^{(3)} + Z^{(1)} Z^{(2)} \right\rangle = 2$$

Stabilizing operators for cluster states

- Stabilizing operators for an *N*-qubit cluster state

$$S_1 = X^{(1)} Z^{(2)},$$

$$S_k = Z^{(k-1)} X^{(k)} Z^{(k+1)}; \quad k = 2, 3, \ldots, N-1,$$

$$S_N = Z^{(N-1)} X^{(N)}.$$

- For separable states

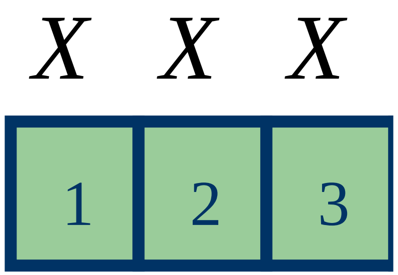
$$\left\langle S_k + S_{k+1} \right\rangle \leq 1$$

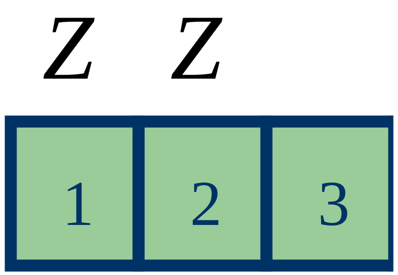
How to measure the conditions?

- Advantage of our conditions that they are easy to measure *locally*

$$\left\langle X^{(1)}X^{(2)}X^{(3)} + Z^{(1)}Z^{(2)} \right\rangle \leq 1$$

- *Two local measurement settings are needed*





Noise tolerance

- In an experiment the GHZ state is never prepared perfectly

$$\rho = (1 - p_{noise}) \left| GHZ_3 \right\rangle \left\langle GHZ_3 \right| + p_{noise} \rho_{completely\_mixed}$$

- For each entanglement condition there is a noise limit. For a noise larger than this limit the GHZ state is not detected as entangled.

Noise tolerance II

- Let us take the condition

$$\left\langle S_1 + S_2 \right\rangle = \left\langle X^{(1)} X^{(2)} X^{(3)} + Z^{(1)} Z^{(2)} \right\rangle \leq 1$$

Noise is tolerated if $p_{noise} < 1/2$

- Better condition, tolerating noise if $p_{noise} < 2/3$

$$\left\langle S_1 + S_2 + S_1 S_2 \right\rangle =$$

$$\left\langle X^{(1)} X^{(2)} X^{(3)} + Z^{(1)} Z^{(2)} - Y^{(1)} Y^{(2)} X^{(3)} \right\rangle \leq 1$$

Genuine multi-qubit entanglement

- Genuine three-qubit entanglement

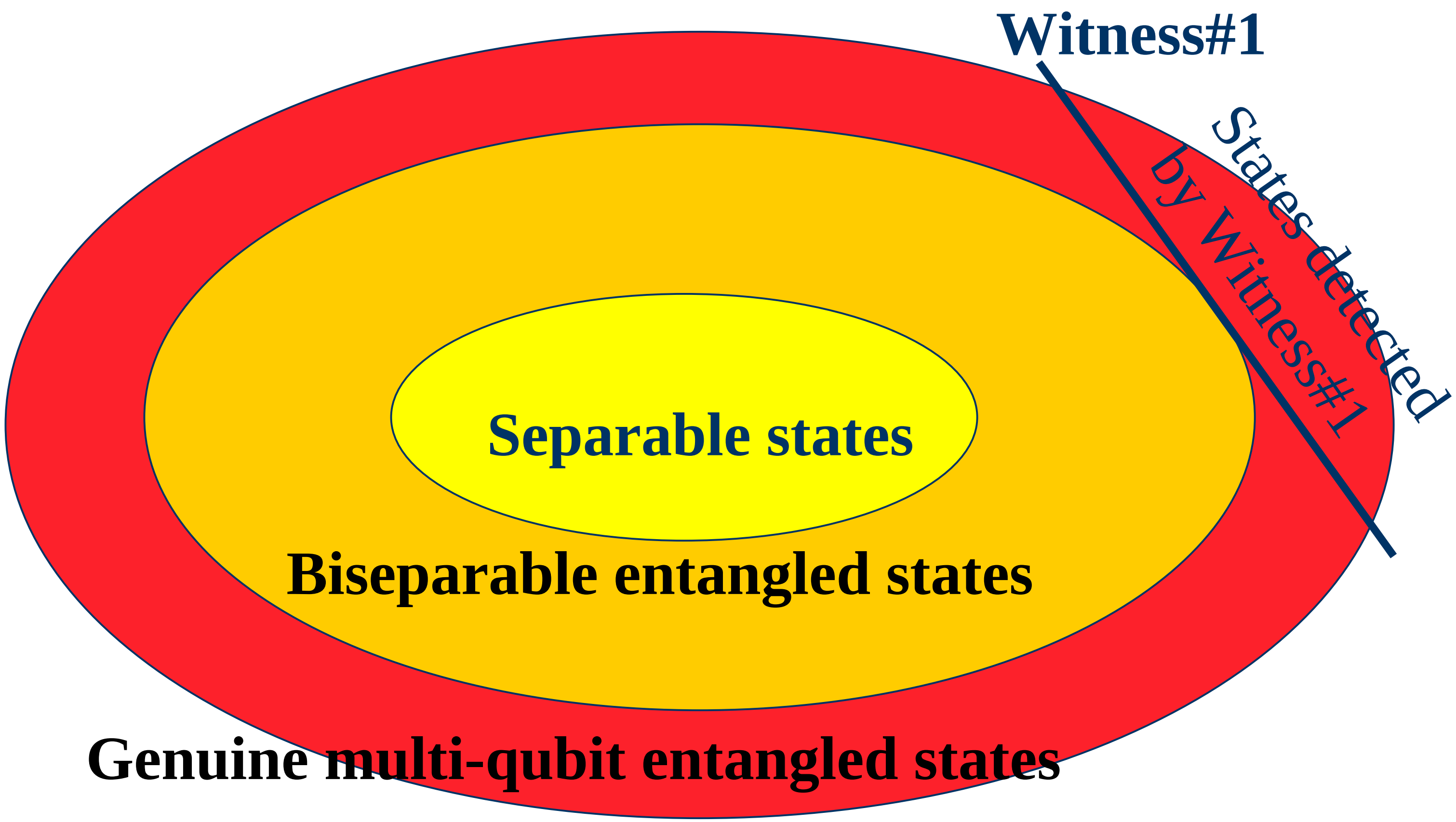
$$|000\rangle + |111\rangle$$

- Biseparable entanglement

$$|001\rangle + |111\rangle =$$
$$\left(|00\rangle + |11\rangle\right)|1\rangle$$

- A mixed entangled state is *biseparable* if it is the mixture of biseparabe states (of possibly different partitions).

Entanglement witnesses for detecting genuine multi-qubit entanglement



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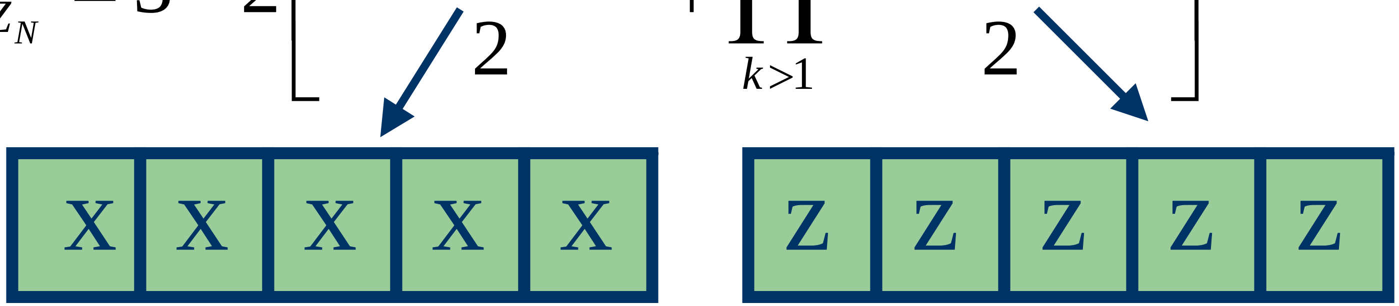
Witnesses for detecting genuine multi-qubit entanglement

$$W_{GHZ_N} = 3 - 2\left[\frac{1+S_1^{(GHZ_N)}}{2} + \prod_{k>1}\frac{1+S_k^{(GHZ_N)}}{2}\right]$$



- Advantage: only the minimal two measurement settings are needed, no exponential increase with the number of qubits!
- Similar ideas can be used to estimate the fidelity.

Conditions based on uncertainty relations

- Based on the ideas of Gühne PRL 92, 117903 for biseparable states of the type (1)(23) we have

$$\left\langle X^{(1)}X^{(2)}X^{(3)} + Z^{(1)}Z^{(2)} \right\rangle +$$

$$\frac{1}{2}\left(\left\langle X^{(1)} + X^{(2)}X^{(3)} \right\rangle^2 + \left\langle Z^{(1)} + Z^{(2)} \right\rangle^2\right) \leq 1$$

- Nonlinear terms: play the role of “refinement“

Conditions based on entropic uncertainty relations

- Based on the ideas of [Gühne & Lewenstein PRA 70, 023316] for biseparable states we have,

$$\sum_{k=1}^{N} H(S_k) \leq \ln 2$$

where *H(A)* is the Shannon entropy of the outcomes after measuring operator *A*.

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Summary

- We constructed entanglement conditions with stabilizing operators.
- Our conditions detect entangled states close to GHZ and cluster states.
- Home page: http://www.mpq.mpg.de/Theorygroup/CIRAC/people/toth
- *************** THANK YOU!!! *************