

Entanglement and Permutational Symmetry

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We study the separability of permutationally symmetric quantum states. We show that for bipartite symmetric systems most of the relevant entanglement criteria coincide. However, we provide a method to generate examples of bound entangled states in symmetric systems, for the bipartite and the multipartite case. These states shed some new light on the nature of bound entanglement.

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Entanglement is a central phenomenon of quantum mechanics and plays a key role in quantum-information processing applications such as quantum teleportation and quantum cryptography [1]. Therefore, entanglement appears as a natural goal of many recent experiments aiming to create various quantum states with photons, trapped ions, or cold atoms in optical lattices. While being at the center of attention, to decide whether a quantum state is entangled or not is still an unsolved problem. There are numerous criteria for the detection of entanglement, but no general solution has been found [1].

Symmetry is another central concept in quantum mechanics [2]. Typically, the presence of certain symmetries simplifies the solution of tasks like the calculation of atomic spectra or finding the ground state of a given spin model. Symmetries are also useful in quantum-information theory: For instance, if a multiparticle quantum state is invariant under the same change of the basis at all parties (i.e., invariant under local unitary transformations of the type $U_{\text{tot}} = U \otimes U \otimes \dots \otimes U$), this symmetry can be used to study the existence of local hidden variable models [3], to determine its entanglement properties [4] or to simplify the calculation of entanglement measures [5].

In this Letter, we investigate to which extent symmetry under permutation of the particles simplifies the separability problem. In general, a quantum state ϱ is called separable, if it can be written as $\varrho = \sum_k p_k \varrho_k^A \otimes \varrho_k^B$, where the p_k form a probability distribution. There are several necessary criteria for a state to be separable. The most famous one is the criterion of the positivity of the partial transposition (PPT), which states that for a separable $\varrho = \sum_{i,j,k,l} \varrho_{ij,kl} |i\rangle\langle j| \otimes |k\rangle\langle l|$ the partially transposed state $\varrho^{TA} = \sum_{i,j,k,l} \varrho_{ji,kl} |i\rangle\langle j| \otimes |k\rangle\langle l|$ has no negative eigenvalues [6]. This criterion is necessary and sufficient only for small systems (2×2 and 2×3), while for other dimensions some entangled states escape the detection [7]. These states are then bound entangled, which means that no pure state

entanglement can be distilled from them. While bound entangled states are difficult to construct, they play an important role in quantum-information theory, as they are at the heart of some open problems in quantum-information theory [7,8]. Apart from the PPT criterion, several other strong separability criteria exist [9–14], which can detect some states where the PPT criterion fails. Also for symmetric states, some special separability criteria have been proposed [15–18].

We will show that for states that are symmetric under a permutation of the particles, most of the relevant known separability criteria coincide. However, we present examples of bound entangled symmetric states. These states form therefore a challenge for the derivation of new separability criteria. Moreover, these states shed new light on the phenomenon of bound entanglement, as it has been suggested that symmetry and bound entanglement are contradicting notions [19–21]. Finally, we present symmetric multipartite bound entangled states, which are nevertheless genuine multipartite entangled.

We first consider two d -dimensional quantum systems. We examine two types of permutational symmetries, denoting the corresponding sets by \mathcal{I} and \mathcal{S} : (i) We call a state *permutationally invariant* (or just invariant, $\varrho \in \mathcal{I}$) if ϱ is invariant under exchanging the particles. This can be formalized by using the flip operator $F = \sum_{ij} |ij\rangle\langle ji|$ as $F\varrho F = \varrho$. The reduced state of two randomly chosen particles of a larger ensemble has this symmetry. (ii) We call a state *symmetric* ($\varrho \in \mathcal{S}$) if it acts on the symmetric subspace only. This space is spanned by the basis vectors $|\phi_{kl}^+\rangle := (|k\rangle|l\rangle + |l\rangle|k\rangle)/\sqrt{2}$ for $k \neq l$ and $|\psi_k\rangle := |k\rangle|k\rangle$. The projector P_s onto this space can be written as $P_s = (1 + F)/2$. This implies that for symmetric states by definition $P_s\varrho P_s = \varrho P_s = \varrho$ and $\varrho F = F\varrho = \varrho$. This is the state space of two d -state bosons.

Clearly, we have $\mathcal{S} \subset \mathcal{I}$. For a basis state of the antisymmetric subspace $|\phi_{kl}^-\rangle := (|k\rangle|l\rangle - |l\rangle|k\rangle)/\sqrt{2}$, we

have, $|\phi_{kl}^-\rangle\langle\phi_{kl}^-| \in I$, however, $|\phi_{kl}^-\rangle\langle\phi_{kl}^-| \notin \mathcal{S}$. Our main tool for the investigation of entanglement criteria is an expectation value matrix of a bipartite quantum state. This matrix has the entries

$$\eta_{kl}(\varrho) := \langle M_k \otimes M_l \rangle_{\varrho}, \quad (1)$$

where M_k 's are local orthogonal observables for both parties, satisfying $\text{Tr}(M_k M_l) = \delta_{kl}$ [22]. We can directly formulate our first main result:

Observation 1.—Let $\varrho \in \mathcal{S}$ be a symmetric state. Then the following separability criteria are equivalent: (i) $\eta \geq 0$, or, equivalently $\langle A \otimes A \rangle \geq 0$ for all observables A [17]. (ii) ϱ has a positive partial transpose, $\varrho^{T_A} \geq 0$ [6]. (iii) ϱ satisfies the CCNR criterion, $\|R(\varrho)\|_1 \leq 1$, where $R(\varrho)$ denotes the realignment map and $\|\dots\|_1$ denotes the trace norm [9,10]. (iv) The correlation matrix, defined via the matrix elements as

$$C_{kl} := \langle M_k \otimes M_l \rangle - \langle M_k \otimes \mathbb{1} \rangle \langle \mathbb{1} \otimes M_l \rangle \quad (2)$$

is positive semidefinite [15]. (v) The state satisfies several variants of the covariance matrix criterion, e.g., $\|C\|_1^2 \leq [1 - \text{Tr}(\varrho_A^2)][1 - \text{Tr}(\varrho_B^2)]$ or $2 \sum |C_{ii}| \leq [1 - \text{Tr}(\varrho_A^2)] + [1 - \text{Tr}(\varrho_B^2)]$ [11,12]. These criteria are for general states strictly stronger than the CCNR criterion.

Here, separability criteria are formulated as conditions that a separable state has to fulfill, and violation implies entanglement of the state. Further, we call two separability criteria equivalent, if a state that is detected by the first criterion is also detected by the second one and vice versa. Note that the criteria (i) and (iv) are criteria specifically for symmetric states, while the others also work for generic separable states. During the proof of this theorem, we will see that several of the equivalences also hold for permutationally invariant states.

Proof.—For invariant states, η is a real symmetric matrix. It can be diagonalized by an orthogonal matrix O . The resulting diagonal matrix $\{\Lambda_k\}$ is the correlation matrix corresponding to the observables $M'_k = \sum O_{kl} M_l$. Hence, any invariant state can be written as

$$\varrho = \sum_k \Lambda_k M'_k \otimes M'_k, \quad (3)$$

where M'_k are pairwise orthogonal observables. This is almost the Schmidt decomposition of the matrix ϱ , with the only difference that Λ_k (which are the eigenvalues of η) can also be negative. Let us compute $\text{Tr}(\eta) = \sum_k \Lambda_k = \langle \sum_k M'_k \otimes M'_k \rangle$. We can use that $\sum_k M'_k \otimes M'_k = F$, where F is again the flip operator [23]. Hence, $-1 \leq \sum_k \Lambda_k \leq 1$ for invariant states and $\sum_k \Lambda_k = 1$ for symmetric states.

Now we can show the first equivalences. Let us start with the CCNR criterion. It states that if ϱ is separable, then $\|R(\varrho)\|_1 \leq 1$, where $\|X\|_1 = \text{Tr}(\sqrt{XX^\dagger})$ is the trace norm and the realigned density matrix is $R(\varrho_{ij,kl}) = \varrho_{ik,jl}$ [10]. As noted in Ref. [24], this just means that if $\|(\varrho F)^{T_A}\|_1 > 1$ then ϱ is entangled. Since for symmetric states $\varrho F = \varrho$, this condition is equivalent to $\|\varrho^{T_A}\|_1 > 1$. This is just the

PPT criterion, since we have $\text{Tr}(\varrho^{T_A}) = 1$ and the condition $\|\varrho^{T_A}\|_1 > 1$ signals the presence of negative eigenvalues.

On the other hand, the realignment criterion can be reformulated as follows: If $\sum_k |\Lambda_k| > 1$ in the Schmidt decomposition [Eq. (3)], then the quantum state is entangled [9]. For symmetric states this is equivalent to $\Lambda_k < 0$ for some k . But then $\langle M'_k \otimes M'_k \rangle < 0$ and η has a negative eigenvalue. For invariant states $\Lambda_k < 0$ is necessary but not sufficient for violating the CCNR criterion. This proves the equivalence of (i),(ii), and (iii).

Now we show that $C \geq 0 \Leftrightarrow \eta \geq 0$. The direction “ \Rightarrow ” is trivial, since for invariant states the matrix $\langle M_k \otimes \mathbb{1} \rangle \times \langle \mathbb{1} \otimes M_l \rangle$ is a projector and hence positive. On the other hand, if we make for a given state the special choice of observables $Q_k = M_k - \langle M_k \rangle$ we just have $C(M_k) = \eta(Q_k)$, implying the other direction. Note that here the invariance of the state guaranteed that the Q_k are the same for both parties.

Finally, let us turn to the discussion of (v). If ϱ is symmetric, the fact that C is positive semidefinite gives $\|C\|_1 = \text{Tr}(C) = \sum \Lambda_k - \sum_k \text{Tr}(\varrho_A M'_k)^2 = 1 - \text{Tr}(\varrho_A^2)$ and similarly, $\sum_i |C_{ii}| = \sum_i C_{ii} = 1 - \text{Tr}(\varrho_A^2)$. Hence, a state fulfilling (iv) fulfills also both criteria in (v). On the other hand, a state violating (iv) must also violate the conditions in (v), as they are strictly stronger than the CCNR criterion [25]. ■

In this proof, sometimes only the permutational invariance was used, hence we can state for invariant states:

Observation 2.—For invariant states, the separability criteria (i), (iv), and (v) are equivalent.

Given the equivalence of the above criteria, it is an interesting question to ask whether there are any entangled symmetric states that escape the detection by these criteria. These states are then PPT and hence bound entangled. Moreover, such states are a challenge for the derivation of new separability criteria, as most of the standard criteria fail. We will present now two methods for constructing such states.

The literature has already examples for invariant bound entangled states: In Ref. [14] Breuer presented, for even $d \geq 4$, a single parameter family of bound entangled states that are I symmetric, namely,

$$\varrho_B = \lambda |\Psi_0^d\rangle\langle\Psi_0^d| + (1 - \lambda) \Pi_s^d. \quad (4)$$

The state is shown to be entangled for $0 < \lambda \leq 1$ while it is PPT for $0 \leq \lambda \leq 1/(d+2)$. Here $|\Psi_0\rangle$ is the singlet state [for the $d=4$ case it is $|\Psi_0\rangle = (|03\rangle - |12\rangle + |21\rangle - |30\rangle)/2$] and Π_s is the normalized projector to the symmetric subspace, $\Pi_s^d = P_s/[d(d+1)/2]$ [26].

The first idea to construct bound entangled states with I - or \mathcal{S} -symmetry is to embed a low-dimensional entangled state into a higher dimensional Hilbert space, such that it becomes symmetric, while it remains entangled. To see a first example, consider a general bound entangled state ϱ_{AB} on \mathcal{H}_{AB} and add to each party a two-dimensional Hilbert space $\mathcal{H}_{A'}$ and $\mathcal{H}_{B'}$. Then one can consider the state

$$\tilde{\varrho} = \frac{1}{2}[|10\rangle\langle 10|_{A'B'} \otimes \varrho_{AB} + |01\rangle\langle 01|_{A'B'} \otimes (F\varrho_{AB}F)]. \quad (5)$$

If ϱ acts on a $d \times d$ system then $\tilde{\varrho}$ acts on a system of size $2d \times 2d$. Obviously, $\tilde{\varrho}$ is an invariant state. If ϱ is entangled, then $\tilde{\varrho}$ is entangled, too, as one can obtain ϱ from $\tilde{\varrho}$ by a local measurement on the ancilla qubits A' and B' . Moreover, if ϱ is PPT then $\tilde{\varrho}$ is also PPT. Substituting the various nonsymmetric bound entangled states available in the literature for ϱ in Eq. (5) gives invariant bound entangled states.

With a similar method, one can generate a symmetric bound entangled state. Starting from Eq. (4) we consider the state

$$\hat{\varrho} = \lambda \Pi_a^D \otimes |\Psi_0^d\rangle\langle \Psi_0^d| + (1 - \lambda) \Pi_s^D \otimes \Pi_s^d. \quad (6)$$

Here, Π_a^D and Π_s^D are appropriately normalized projectors to the two-qudit symmetric/antisymmetric subspace with dimension D , e.g., $\Pi_a^D = P_a/[D(D-1)/2]$ with $P_a = \mathbb{1} - P_s$. This guarantees that $\hat{\varrho}$ is symmetric. Again, if the original system is of dimension $d \times d$ then the system of $\hat{\varrho}$ is of dimension $dD \times dD$. Since ϱ_B is the reduced state of $\hat{\varrho}$, if the first is entangled, then the second is also entangled. However, it is not clear from the beginning that if ϱ_B is PPT then $\hat{\varrho}$ is also PPT, since Π_a^D is entangled. For $D = 2$ and $N = 4$, however, numerical calculation shows that $\hat{\varrho}$ is PPT for $\lambda < 0.062$. This provides an example of an \mathcal{S} symmetric (and invariant) bound entangled state of size 8×8 . Note that this state represents an explicit counterexample to Ref. [21]. There, it has been suggested that an invariant state with $\eta \geq 0$ has to be of the form $\varrho = \sum_k p_k \varrho_k \otimes \varrho_k$, which implies that it is separable [27]. Also, in Ref. [19] it has been found that the nondistillability of entangled quantum states may be connected to the asymmetry of quantum correlations in that state. While this may be valid for many examples, our results demonstrate that there is not a strict rule connecting the two phenomena.

Finally, we show a simple method for constructing symmetric bipartite bound entangled states numerically. We first generate an N -qubit symmetric state, that is, a state of the symmetric subspace. We consider even N . It is known that such a state is either separable with respect to all bipartitions or it is entangled with respect to all bipartitions [28]. Thus any state that is PPT with respect to the $\frac{N}{2}:\frac{N}{2}$ partition while NPT with respect to some other partition is bound entangled with respect to the $\frac{N}{2}:\frac{N}{2}$ partition. Since the state is symmetric, it can straightforwardly be mapped to a $(\frac{N}{2} + 1) \times (\frac{N}{2} + 1)$ bipartite symmetric state [29].

To obtain such a multiqubit state, one has to first generate an initial random state ϱ that is PPT with respect to the $\frac{N}{2}:\frac{N}{2}$ partition. Reference [30] describes how to get a random density matrix with a uniform distribution according to the Hilbert-Schmidt measure. Then, we compute the minimum nonzero eigenvalue of the partial transpose of ϱ with respect to all other partitions $\lambda_{\min}(\varrho) := \min_k \min_l \lambda_l(\varrho^{T_{l_k}})$. Here l_k describes which

qubits to transpose for the partition k . If $\lambda_{\min}(\varrho) < 0$ then the state is bound entangled with respect to the $\frac{N}{2}:\frac{N}{2}$ partition. If it is non-negative then we start an optimization process for decreasing this quantity. The zero eigenvalues due to the nonmaximal rank of symmetric states are excluded from the minimization, otherwise one always gets $\lambda_{\min} \leq 0$.

We generate another random density matrix $\Delta\varrho$, and check the properties of $\varrho' = (1 - \varepsilon)\varrho + \varepsilon\Delta\varrho$, where $0 < \varepsilon < 1$ is a small constant. If ϱ' is still PPT with respect to the $\frac{N}{2}:\frac{N}{2}$ partition and $\lambda_{\min}(\varrho') < \lambda_{\min}(\varrho)$ then we use ϱ' as our new random initial state ϱ . If this is not the case, we keep the original ϱ . Repeating this procedure, we obtained a four-qubit symmetric state this way

$$\varrho_{BE4} = \text{diag}(0.22, 0.176, 0.167, 0.254, 0.183) - 0.059R,$$

where $R := |3\rangle\langle 0| + |0\rangle\langle 3|$. The basis states are $|0\rangle := |0000\rangle$, $|1\rangle := \text{sym}(|1000\rangle)$, $|2\rangle := \text{sym}(|1100\rangle)$, ... where $\text{sym}(A)$ denotes an equal superposition of all permutations of A . The state is bound entangled with respect to the 2:2 partition. This corresponds to a 3×3 bipartite symmetric bound entangled state [29], demonstrating the simplest possible symmetric bound entangled state.

Our method can be straightforwardly generalized to create multipartite bound entangled states of the symmetric subspace, such that *all* bipartitions are PPT (“fully PPT states”). Then, however, a new separability criterion must be used, different from the PPT criterion. The PPT symmetric extension of Doherty *et al.* [13] seems to be ideal for our case. For symmetric states, it can be formulated as follows: We define the PPT symmetric extension of an N -qubit state ϱ_N as a symmetric M -qubit state, ϱ_M such that $\varrho_N = \text{Tr}_{N+1, N+2, \dots, M}(\varrho_M)$, and all bipartitions of ϱ_M are PPT. If there is an M for which such an extension does not exist then our state is entangled. Semidefinite programming makes it possible to look for such an extension. Note that the two density matrices can be efficiently stored as $(M+1) \times (M+1)$ and $(N+1) \times (N+1)$ matrices, respectively, in the symmetric basis, making it possible to look for very large extensions or examine large states [31]. Moreover, similarly to the algorithm described in the previous paragraph, it is possible to design a simple random search that, starting from fully PPT random nonentangled states, leads to PPT states without an extension. We found such a state for five qubits

$$\varrho_{BE5} = \text{diag}(0.17, 0.174, 0.153, 0.182, 0.147, 0.174) - \mathcal{Q},$$

where $\mathcal{Q} := 0.0137(|4\rangle\langle 0| + |0\rangle\langle 4|)$, and the basis states of the symmetric system are $|0\rangle, |1\rangle, \dots, |4\rangle$. We also found such a state for six qubits [32].

These multiqubit states are by construction genuine multipartite entangled [28,33]. This finding is quite peculiar: Genuine multipartite entanglement is considered in a sense a strong type of entanglement, while local states or states with PPT bipartitions are considered weakly entangled. It is interesting to relate this to the Peres conjec-

ture, stating that fully PPT states cannot violate a Bell inequality [34]. If this is true, then we presented genuine multiqubit states that are local. So far, such states have been known only for the three-qubit case [35].

In summary, we have discussed entanglement in symmetric systems. We showed that for states that are in the symmetric subspace several relevant entanglement conditions, especially the PPT criterion, the CCNR criterion, and the criterion based on covariance matrices, coincide. We showed the existence of symmetric bound entangled states, in particular, a 3×3 , five-qubit and six-qubit symmetric PPT entangled states.

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