Detection of multipartite entanglement in the vicinity of symmetric Dicke states

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I present methods for detecting entanglement around symmetric Dicke states. In particular, I consider N-qubit symmetric Dicke states with $N/2$ excitations. I show that for large N these states have the smallest overlap possible with states without genuine multipartite entanglement. Thus these states are particularly well suited for the experimental examination of multipartite entanglement. I present fidelity-based entanglement witness operators for detecting multipartite entanglement around these states. I then consider entanglement criteria, somewhat similar to the spin squeezing criterion, based on the moments or variances of the collective spin operators. Surprisingly, these criteria are based on an upper bound for variances for separable states. I present criteria detecting entanglement in general and criteria detecting only genuine multipartite entanglement. The collective operator measured for this criteria is an important physical quantity: Its expectation value essentially gives the intensity of the radiation when a coherent atomic cloud emits light.

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1. INTRODUCTION

The nonclassical effects of quantum mechanics have already been studied theoretically for more than 50 years.[1] Which quantum states can lead to phenomena that are strikingly nonclassical? Which quantum states are useful for quantum information processing applications? The answers to these questions lead to the definition of separability, entanglement,[2] and multipartite entanglement.[3,4]

In the past decade, with the rapid development of quantum control[5] it has become possible to examine the nonclassicality of quantum mechanics experimentally by creating multiqubit quantum states of photons,[6–13] trapped ions,[3] and cold atoms on an optical lattice.[14] The first multiqubit experiments concentrated on Greenberger–Horne–Zeilinger[15] (GHZ) states. As maximally entangled multiqubit states, they are intensively studied and have been realized in numerous experiments.[3,6–8] Other quantum states targeted in experiments due to their interesting properties are, for example, cluster states[11–13,16,17] and many-body singlet states.[18]

In this paper we will discuss some of the advantages of using Dicke states[19] to study the nonclassical phenomena of quantum mechanics. In his seminal paper, Dicke[19] considered the spontaneous emission of light by a cloud of two-state atoms that are coupled to the electromagnetic field as electric dipoles. He found that when the cloud acts as a coherent quantum system, the maximal light intensity is roughly proportional to the square of the number of atoms, which he called superradiance. The highly correlated Dicke states, defined to describe the system above, are the simultaneous eigenstates of the collective angular momentum J and its z component J_z. In a typical many-qubit experiment, in which the qubits cannot be individually accessed, both the initial state and the dynamics are symmetric under the permutation of qubits. Thus in this paper we will consider only symmetric Dicke states. These are also the states with maximal J. An N-qubit symmetric Dicke state with m excitations is defined as[20]

$$|m,N\rangle := \binom{N}{m}^{-1/2} \sum_k P_k(|1_1,1_2,\ldots,1_m,0_{m+1},\ldots,0_N\rangle), \tag{1}$$

where $\{P_k\}$ is the set of all distinct permutations of the spins. $|1,N\rangle$ is the well-known W state.

Several proposals have been presented in the literature for the experimental creation of Dicke states. In Ref. 21 a scheme is considered for creating Dicke states in trapped ions using an adiabatic process. A method for the realization of arbitrary superposition of symmetric Dicke states by detecting the photons leaving a cavity is described in Ref. 22. A novel scheme has been proposed for obtaining Dicke states based on creating closed subspaces for the quantum dynamics of an ion chain.[23] Other proposals are described, for example, in Refs. 24–27.

On the experimental side, I have to mention that a three-qubit W state has been created in a photonic system.[10,28,29] Also, an eight-qubit W state has been prepared with trapped ions.[30] Recently, a four-qubit Dicke state with two excitations has been created in a photonic system.[31] It turned out that this is one of the quantum states that can be obtained in a photonic experiment with good fidelity. Future experiments will most certainly lead to creation of Dicke states with multiple excitations in other physical systems. At this point it is important to ask the question: Are such states more useful than others from the point of view of quantum information processing? In Ref. 31 it has already been discussed that the Dicke state prepared in the experiment is useful for telecloning.

In this paper I demonstrate that Dicke states with multiple excitations are also good candidates for the experimental examination of genuine multipartite entanglement.[4] In particular, I discuss how to detect entanglement close to $|N/2,N\rangle$, i.e., an N-qubit symmetric Dicke state with $N/2$ excitations. We will find that, similar to GHZ[15] and cluster states,[16] for large N such states have the smallest overlap possible with states without multipartite entanglement.

In the second part of the paper, entanglement detection schemes requiring only collective measurements are discussed.[32–38] Entanglement detection with collective measurements is important since in many experiments the qubits cannot be accessed individually. Even if the qubits could be individually accessed, our measurement schemes are still useful since they need small experimental effort.[31] The schemes presented are based on an upper bound on the variances of collective observables for separable states. Any state violating this bound is detected as entangled. I present schemes for entanglement detection in general and also schemes for detecting only genuine multipartite entanglement.

$|N/2,N\rangle$ is exactly the quantum state for which Dicke found that the superradiance is the strongest[19] for even N. I will show that if my schemes are applied to a system described in Dicke's original paper[19] then the measurement of the collective observables of the scheme is essentially equivalent to the measurement of light intensity emitted by the atoms.

The paper is organized as follows. In Section 2 I show that for a fidelity-based detection of multipartite entanglement, the required fidelity is low for this state. In Section 3 I discuss entanglement detection with collective observables close to the state $|N/2,N\rangle$. In Appendix A I present some calculations for Section 2.

2. FIDELITY-BASED ENTANGLEMENT CRITERIA

Before starting our main discussion, let us first review the basic terminology of the field. An N-qubit state is called fully separable if its density matrix can be written as the mixture of product states,

$$\rho = \sum_l p_l \rho_l^{(1)} \otimes \rho_l^{(2)} \otimes \cdots \otimes \rho_l^{(N)}, \tag{2}$$

where $\Sigma_l p_l = 1$ and $p_l > 0$. Otherwise the state is called entangled. Quantum optics experiments aim to create entangled states, since these are the quantum states that lead to phenomena very different from classical physics.[5]

In a multiqubit experiment it is important to detect genuine multiqubit entanglement.[10] We have to show that all the qubits are entangled with each other, not only some of them. An example of the latter case is a state of the form

$$|\Phi\rangle = |\Phi_{1\ldots m}\rangle \otimes |\Phi_{m+1\ldots N}\rangle. \tag{3}$$

Here $|\Phi_{1\ldots m}\rangle$ denotes the state of the first m qubits while $|\Phi_{m+1\ldots N}\rangle$ describes the state of the remaining qubits. Note that the state given by Eq. (3) might be entangled, but it is separable with respect to the partition $(1,2,\ldots,m)(m+1,m+2,\ldots,N)$. Such states are called biseparable[4] and can be created from product states such that two groups of qubits do not interact. These concepts can be extended to mixed states. A mixed state is biseparable if it can be created by mixing biseparable pure states of the form of Eq. (3). Note that we get mixed biseparable states even when mixing pure biseparable states that are separable with respect to different partitions [e.g., when mixing $(|00\rangle+|11\rangle)|0\rangle/\sqrt{2}$ and $|0\rangle(|00\rangle+|11\rangle)/\sqrt{2}$]. An N-qubit state is said to have genuine N-partite entanglement if it is not biseparable.

Now we will present conditions for the detection of genuine multipartite entanglement. These will be criteria based on entanglement witness operators.[39–46] In other words, these are criteria that are linear in operator expectation values.[47] On the basis of Ref. 10 we know that for biseparable states ρ,

$$\mathrm{Tr}(\rho|\Psi\rangle\langle\Psi|) \le C_\Psi. \tag{4}$$

Here $|\Psi\rangle$ is a multiqubit entangled state and C_Ψ is the square of the maximal overlap of $|\Psi\rangle$ with biseparable states[10]:

$$C_\Psi := \max_{\phi\in\mathcal{B}} |\langle\Psi|\phi\rangle|^2, \tag{5}$$

where $\mathcal{B}$ denotes the set of biseparable pure states. Any state ρ violating inequality (4) is necessarily genuine multipartite entangled. The bound in inequality (4) is sharp, that is, it is the lowest possible bound. Computing Eq. (5) seems to be a complicated optimization problem. Fortunately, it turns out that C_Ψ equals the square of the maximum of the Schmidt coefficients of $|\Psi\rangle$ with respect to all bipartitions.[10] Thus C_Ψ can be determined easily, without the need for multivariable optimization.

The use of criteria of the type in inequality (4) is as follows. Let us say that in an experiment one aims to prepare the state $|\Psi\rangle$. This preparation is not perfect; however, one might still expect that the state prepared in the experiment be close to $|\Psi\rangle$. Thus a fidelity-based entanglement criterion of the type in inequality (4) can be used to detect its entanglement. The smaller the required minimal fidelity C_Ψ, the better the criterion from a practical point of view.

Now we present criteria of the form of inequality (4) for detecting entanglement around symmetric Dicke states.

Theorem 1. For biseparable quantum states ρ,

$$\mathrm{Tr}(\rho|N/2,N\rangle\langle N/2,N|) \le \frac{1}{2}\frac{N}{N-1} =: C_{N/2,N}. \tag{6}$$

This condition detects entanglement close to an N-qubit symmetric Dicke state with $N/2$ excitations. Here N is assumed to be even.

Proof. The Schmidt decomposition of $|m,N\rangle$ according to the partition $(1,2,\ldots,N_1)(N_1+1,N_1+2,\ldots,N)$ is[20]

$$|m,N\rangle = \sum_k \lambda_k |k,N_1\rangle \otimes |m-k,N-N_1\rangle, \tag{7}$$

where the Schmidt coefficients are

$$\lambda_k = \binom{N}{m}^{-1/2} \binom{N_1}{k}^{1/2} \binom{N-N_1}{m-k}^{1/2}. \tag{8}$$

We do not have to consider other partitions due to the permutational symmetry of our Dicke states. For $|N/2,N\rangle$ we have $m=N/2$. Now we use that

$$\binom{N_1}{k}\binom{N-N_1}{\frac{N}{2}-k} \le \binom{2}{1}\binom{N-2}{\frac{N}{2}-1}. \tag{9}$$

The proof of inequality (9) can be found in Appendix A. Thus we find that the maximal Schmidt coefficient can be obtained for $N_1=2$ and $k=1$. For $N_1=2$ we obtain $\lambda_1^2 = N(N-1)/2$.

Thus we find that $C_{N/2,N} \approx 1/2$ for large N. This makes the detection of multipartite entanglement around the state $|N/2,N\rangle$ relatively easy. This property is quite remarkable: Up to now only GHZ,[15] cluster,[16] and graph states[48] have been known to have $C=1/2$.[49,50]

Connected to the previous paragraph, it is important to check how much our entanglement criterion is robust against noise. To see this, let us consider a $|N/2,N\rangle$ state mixed with white noise:

$$\varrho(p) = p_{\text{noise}} \frac{\mathbb{1}}{2^N} + (1-p_{\text{noise}})|N/2,N\rangle\langle N/2,N|, \tag{10}$$

where p_{noise} is the ratio of noise. Our criterion is very robust: It detects a state of the form of Eq. (10) as true multipartite entangled if

$$p_{\text{noise}} < \frac{1}{2}\left[\frac{N-2}{(N-1)(1-2^{-N})}\right]. \tag{11}$$

For large N we have $p_{\text{noise}} < 1/2$.

Note that the situation is different for a W state. A condition that can be obtained for detecting genuine multipartite entanglement around a W state is[10,30]

$$\mathrm{Tr}(\rho|1,N\rangle\langle 1,N|) \le \frac{N-1}{N} =: C_{1,N}. \tag{12}$$

Any state violating this condition is multipartite entangled. However, note that with an increasing $N, C_{1,N}$ approaches 1 rapidly. This makes multipartite entanglement detection based on inequality (12) challenging.

3. ENTANGLEMENT DETECTION WITH COLLECTIVE MEASUREMENTS

In Section 2 for Theorem 1 we needed the measurement of the expectation value of $|N/2,N\rangle\langle N/2,N|$. To measure this operator, it must be decomposed into the sum of multiqubit correlation terms of the form $A^{(1)} \otimes A^{(2)} \otimes A^{(3)} \otimes \ldots$,[10,49–54] where $A^{(k)}$ acts on qubit k. For measuring the expectation value of such correlation terms, we must be able to access the qubits individually.

However, in certain physical systems (e.g., optical lattices of bosonic two-state atoms[14]) only the measurement of collective quantities is possible. In this section I present entanglement criteria for detecting entanglement with collective measurements.[32–38] Our entanglement conditions will be built using the collective spin operators

$$J_{x/y/z} := \frac{1}{2}\sum_{k=1}^{N} \sigma_{x/y/z}^{(k)}, \tag{13}$$

where $\sigma_{x/y/z}^{(k)}$ denote Pauli spin matrices acting on qubit k.

Lemma 1. For separable states the maximum of the expression

$$a_x\langle J_x^2\rangle + a_y\langle J_y^2\rangle + a_z\langle J_z^2\rangle + b_x\langle J_x\rangle + b_y\langle J_y\rangle + b_z\langle J_z\rangle, \tag{14}$$

with $a_{x/y/z} \ge 0$ and real $b_{x/y/z}$ is the same as its maximum for translationally invariant product states (i.e., for product states of the form $|\Psi\rangle = |\psi\rangle^{\otimes N}$). In particular, if $b_x = b_y = b_z = 0$, then this expression is bounded from above by

$$B := (a_x + a_y + a_z)\frac{N}{4} + \max(a_x,a_y,a_z)\frac{N}{2}\left(\frac{N}{2} - \frac{1}{2}\right). \tag{15}$$

Proof. Because of the convexity of separable states, it is enough to look for the maximum for pure product states. For technical reasons we consider a mixed product state of the form $\rho = \otimes_{k=1}^{N} \rho^{(k)}$ and use the notation $s_{x/y/z}^{(k)} := \mathrm{Tr}(\rho^{(k)}\sigma_{x/y/z})/2$. Hence we have to maximize

$$f := (a_x + a_y + a_z)N + \sum_{l=x,y,z} a_l \left[\left(\sum_k s_l^{(k)}\right)^2 - \sum_k (s_l^{(k)})^2\right] + b_l \sum_k s_l^{(k)}. \tag{16}$$

Let us consider the constraints

$$\sum_k s_l^{(k)} = K_l \tag{17}$$

for $l=x,y,z$, where K_l are some constants. Note that f can be written as $f = (a_x+a_y+a_z)N + a_x f_x + a_y f_y + a_z f_z$. Now let us first take f_x, that is, the part that depends only on the $s_x^{(k)}$ coordinates. It can be written as

$$f_x = \left(\sum_k s_x^{(k)}\right)^2 - \sum_k (s_x^{(k)})^2 + \alpha \sum_k s_x^{(k)}, \tag{18}$$

where $\alpha_x = b_x/a_x$. We build the constraint of Eq. (17) into our calculation by the substitution

$$s_x^{(N)} = K_x - \sum_{k=1}^{N-1} s_x^{(k)}. \tag{19}$$

Then for any $m<N$ we obtain the derivatives as

$$\frac{\partial f_x}{\partial s_x^{(m)}} = -2s_x^{(m)} + 2\left(K_x - \sum_{k=1}^{N-1} s_x^{(k)}\right). \tag{20}$$

In an extreme point this should be zero. Hence it follows that for all $m<N$,

$$s_x^{(m)} = s_x^{(N)}; \tag{21}$$

thus f_x takes its extremum when all $s_x^{(m)}$ that are equal. Let us now see whether this extreme point is a maximum. For any $m, n < N$,

$$\frac{\partial^2 f_x}{\partial s_x^{(m)} \, \partial s_x^{(n)}} = -2 - 2\delta_{mn}, \tag{22}$$

where δ_{mn} is the Kronecker symbol. It is easy to see that the matrix containing the second-order derivatives is negative definite, thus our extremum is a maximum. It is also a global maximum, since based on Eq. (18) and the constraint of Eq. (17) it is obvious that if any $|s_x^m| \to \infty$ then $f_x \to -\infty$. Similar calculations can be carried out for the part of f depending on the y and z coordinates. We have just proved that for a given $K_{x/y/z}$, f given in Eq. (16) takes its maximum for translationally invariant product states for which $s_{x/y/z}^{(k)} = K_{x/y/z}/N$. We will denote this maximum by $f_{\max}(K_x, K_y, K_z)$.

Let us now look for the K_x, K_y, and K_z for which $f_{\max}$ is maximal. The condition for obtaining physical state is $\Sigma_l (K_l/N)^2 \le 1/4$ where the equality holds for pure product states. We find that $f_{\max}$ is convex, thus it takes its maximum at the boundary of the domain allowed for $K_{x/y/z}$, i.e., it takes its maximum for pure translationally invariant product states. Hence the upper bound of Eq. (15) for f follows.

In general it is hard to find the maximum for an operator expectation value for separable states.[55–58] We have just proved that for operators of the form of expression (14) that are constructed from first and second moments of the angular-momentum coordinates, this problem is easy: It can be reduced to a maximization over states of the form $|\psi\rangle^{\otimes N}$, i.e., to a maximization over three real variables $s_{x/y/z}$. Note that it is not at all clear from the beginning that this simplification is possible. For example, when looking for the minimum of $J_x^2 + J_y^2 + J_z^2$ for pure product states, it turns out that the expression is not minimized by translationally invariant product states. To be more specific, for $N=2$, when we minimize this expression for product states, the minimum is obtained for the state $|1\rangle|-1\rangle$.

Theorem 2. As a special case of the previous criterion, we have that for separable seperates,[59]

$$\langle J_x^2 \rangle + \langle J_y^2 \rangle \le \frac{N}{2}\left(\frac{N}{2} + \frac{1}{2}\right). \tag{23}$$

For even N, the left-hand side is the maximal $N/2(N/2+1)$ only for an N-qubit symmetric Dicke state with $N/2$ excitations. On the basis of Lemma 1, the proof of this theorem is obvious. It can also be seen that the bound in inequality (23) is sharp since a separable state of the form

$$|\Psi_{xy}\rangle := 2^{-N/2}(|0\rangle + |1\rangle e^{i\phi})^{\otimes N} \tag{24}$$

for any real ϕ saturates the bound.

On the basis of inequality (23), it is easy to see that for separable states we also have

$$(\Delta J_x)^2 + (\Delta J_y)^2 \le \frac{N}{2}\left(\frac{N}{2} + \frac{1}{2}\right). \tag{25}$$

Thus $J_{x/y}^2$ could be replaced by the corresponding variances. Any state violating inequality (25) is entangled. Note the curious nature of our criterion: A state is detected as entangled if the uncertainties of the collective spin operators are larger than a bound.

How can we intuitively understand the criterion of inequality (23)? Using the notation $\mathbf{J} = (J_x, J_y, J_z)$, one can rewrite it as[60]

$$\langle \mathbf{J}^2 \rangle - \frac{N}{2}\left(\frac{N}{2} + \frac{1}{2}\right) \le \langle J_z^2 \rangle. \tag{26}$$

For a given $\langle \mathbf{J}^2 \rangle$, to violate inequality (26), $\langle J_z^2 \rangle$ must be sufficiently low. For symmetric states (i.e., for states that could be used to describe two-state bosons) we have $\langle \mathbf{J}^2 \rangle = N/2(N/2+1)$ and inequality (26) turns into the condition

$$\frac{N}{4} \le \langle J_z^2 \rangle. \tag{27}$$

A condition similar to inequality (27) has already been presented for the detection of two-qubit entanglement for symmetric states in Refs. 37 and 38.

The criterion of inequality (23) detects the state of the form of Eq. (10) as entangled if $p_{\text{noise}} < 1/N$. Note that the limit on p_{noise} decreases rapidly with N. Let us now consider a different type of noise:

$$\varrho'(p) = p_{\text{noise}} |\Psi_{xy}\rangle\langle\Psi_{xy}| + (1 - p_{\text{noise}}) |N/2, N\rangle\langle N/2, N|, \tag{28}$$

where Ψ_{xy} is defined in Eq. (24). Then the citerion of inequality (23) detects the state as entangled for any $p_{\text{noise}} < 1$. Thus the usefulness of our criteria depends strongly on the type of noise appearing in an experiment.

Criteria can also be obtained that detect entanglement around other multiqubit Dicke states. For example, the expression[21]

$$\langle J_x^2 \rangle + \langle J_y^2 \rangle - 2m\langle J_z \rangle \tag{29}$$

takes its maximum at a Dicke state $|m + N/2, N\rangle$. The maximum for separable states can be obtained from Lemma 1.

Up to now we have discussed how to detect entanglement with the measurement of collective observables. Now we show that a criterion similar to the one in Theorem 2 can be used to detect genuine multipartite entanglement. Such a criterion has already been presented for three qubits in Ref. 61. For biseparable three-qubit states,

$$\langle J_x^2 \rangle + \langle J_y^2 \rangle \le 2 + \sqrt{5}/2 \approx 3.12. \tag{30}$$

Both the state $|W\rangle = |1,3\rangle$ and the state $|\bar{W}\rangle = |2,3\rangle$ give the maximal 3.75 for the left-hand side of inequality (30).

Now let us look for criteria for larger systems. To proceed, we will need the following:

Lemma 2. For a two-qubit quantum state,

$$\langle M_1\rangle^2 + \langle M_2\rangle^2 + \langle M_3\rangle^2 \le \frac{16}{3}, \tag{31}$$

where

$$M_1 := \sigma_x^{(1)}\sigma_x^{(2)} + \sigma_y^{(1)}\sigma_y^{(2)},$$
$$M_2 := \sigma_x^{(1)} + \sigma_x^{(2)},$$
$$M_3 := \sigma_y^{(1)} + \sigma_y^{(2)}. \tag{32}$$

Proof. The proof is rather technical. Let us consider the vector $v := (\langle M_1\rangle, \langle M_2\rangle, \langle M_3\rangle)$. We want to find an upper bound on $|v|$. We can easily write

$$|v|^2 = \langle M_1\rangle^2 + \langle M_2\rangle^2 + \langle M_3\rangle^2. \tag{33}$$

We have to look for the maximum of this expression for quantum states. The problem is that it is nonlinear in operator expectation values. Because of that we will employ the following equality:

$$|v| = \max_{|n|=1} vn, \tag{34}$$

where n is a real unit vector. The meaning of Eq. (34) is clear: The length of a vector is equal to the maximum of its scalar product with a unit vector. Now the right-hand side of Eq. (34) can be rewritten as

$$|v| = \max_{|n|=1} \langle n_1M_1 + n_2M_2 + n_3M_3\rangle. \tag{35}$$

The advantage of this expression is that it is linear in operator expectation values. The disadvantage is that we have to maximize over n. Now we will find an upper bound on the right-hand side of Eq. (35). We will use the fact that for an operator A the expectation value is bounded as $\langle A\rangle \le \Lambda_{\max}(A)$. Here $\Lambda_{\max}(A)$ denotes the largest eigenvalue of operator A. Thus

$$|v| \le \max_{|n|=1} \Lambda_{\max}(n_1M_1 + n_2M_2 + n_3M_3). \tag{36}$$

The eigenvalues of $(n_1M_1+n_2M_2+n_3M_3)$ can easily be obtained analytically as the function of n_k. They are

$$\lambda_1 = 0,$$
$$\lambda_2 = -2n_1,$$
$$\lambda_{3/4} = n_1 \pm \sqrt{n_1^2 + 4n_2^2 + 4n_3^2}. \tag{37}$$

Assuming $|n|=1$, the eigenvalues given in Eqs. (37) are bounded from above by $\sqrt{16/3}$. Hence, based on inequality (36) we obtain $|v|^2 \le 16/3$ and inequality (31) follows.

Using Lemma 2, we can state the following:

Theorem 3. For a four-qubit biseparable state,

$$\langle J_x^2\rangle + \langle J_y^2\rangle \le \frac{7}{2} + \sqrt{3} \approx 5.23. \tag{38}$$

For the left-hand side of inequality (38) the maximum is 6 and it is obtained uniquely for the $|2,4\rangle$ state.

Proof. First we present the proof for biseparable pure states with a (12) (34) partition. For these, $\langle J_x^2\rangle + \langle J_y^2\rangle = 2 + v_1v_2/2$ where

$$v_1 := (x_1x_2 + y_1y_2, x_1 + x_2, \quad y_1 + y_2, 1),$$
$$v_2 := (1, x_3 + x_4, \quad y_3 + y_4, x_3x_4 + y_3y_4). \tag{39}$$

Here we used the notation $x_1x_2 = \langle \sigma_x^{(1)}\sigma_x^{(2)}\rangle$. Hence a bound can be obtained using the Cauchy–Schwarz inequality as $\langle J_x^2\rangle + \langle J_y^2\rangle \le 2 + |v_1||v_2|/2 \le 31/6 \approx 5.17$, where from Lemma 2, we have $|v_k|^2 \le 16/3$. Note that the upper bound that we just obtained for $\langle J_x^2\rangle + \langle J_y^2\rangle$ is smaller than the bound in inequality (38); thus biseparable pure states with a (12)(34) partition fulfill inequality (38).

Now let us take biseparable states with the partition (12)(34). We will follow similar steps as in the proof of Lemma 2. Let us define the matrices:

$$Q_a := \sigma_a^{(2)} + \sigma_a^{(3)} + \sigma_a^{(4)}; \quad a = x, y,$$
$$R := \sum_{l=x,y} \sigma_l^{(2)}\sigma_l^{(3)} + \sigma_l^{(2)}\sigma_l^{(4)} + \sigma_l^{(3)}\sigma_l^{(4)}. \tag{40}$$

Using these matrices we can write

$$\langle J_x^2\rangle + \langle J_y^2\rangle = 2 + \frac{1}{2}(x_1\langle Q_x\rangle + y_1\langle Q_y\rangle + \langle R\rangle)$$
$$\le 2 + \frac{1}{2} \max_{x_1^2+y_1^2\le 1} \Lambda_{\max}(x_1Q_x + y_1Q_y + R). \tag{41}$$

Again, to find an upper bound we need the eigenvalues of $(x_1Q_x+y_1Q_y+R)$. These are

$$\lambda_{1,2} = -2 + X,$$
$$\lambda_{3,4} = -2 - X,$$
$$\lambda_{5,6} = 2 + X \pm 2\sqrt{1 + X + X^2},$$
$$\lambda_{7,8} = 2 - X \pm 2\sqrt{1 - X + X^2}, \tag{42}$$

where $X = \sqrt{x_1^2 + y_1^2}$. Assuming $|X| \le 1$, the upper bound of the eigenvalues in Eqs. (42) is $3 + 2\sqrt{3}$. Thus, based on Eqs. (41), we obtain inequality (38) for biseparable states with a (1)(234) partition.

Since the measured operators are symmetric under the permutation of qubits, this also proves that inequality (38) holds for any biseparable pure state. Because of the convexity of biseparable states, it also holds for mixed biseparable states.

The criterion of inequality (30) and Theorem 3 have already been used in the experiment with photons described in Ref. 31 for detecting multipartite entanglement in three-qubit and four-qubit systems. Let us now briefly outline how to detect multipartite entanglement for more than four qubits. For many qubits, detecting multipartite entanglement becomes difficult with collective observables, since (i) the robustness to noise is decreasing as the number of qubits are increasing and (ii) it is hard to obtain the bound for biseparable states for an operator expectation value. The first problem can be handled building entanglement criteria that use higher-order moments of the angular-momentum coordinates $J_{x/y}$. This makes the robustness to noise somewhat better. The second problem can be overcome, for example, by using the method applied in Refs. 49 and 50. This makes it possible to find upper bounds for operator expectation values for biseparable states for a large number of qubits.

Finally, let us discuss how our entanglement conditions of inequalities (23), (30), and (38) are connected to superradiance. The left-hand side of inequality (23) is the same expression that appears in Eq. (28) of Dicke's original paper[19] giving the light intensity of the superradiant light during spontaneous emission in a cloud of atoms. To be more precise, the light intensity is $I := I_0\langle J_x^2 + J_y^2 + J_z\rangle$ where I_0 is the radiation rate of one atom in its excited state. The criterion of inequality (23) shows that if $I/I_0\langle J_z\rangle$ is larger than a bound, then the system is entangled. We can also see that there are separable states [e.g., the state presented in Eq. (24)] for which the light intensity scales roughly with the square of the number of qubits.

4. CONCLUSION

I have presented several methods for detecting entanglement in the vicinity of symmetric Dicke states with multiple excitations. In particular, I focused on N-qubit symmetric Dicke states with $N/2$ excitations. I showed that they are well suited for experiments aiming to create and detect multipartite entanglement. I presented fidelity-based criteria for detecting genuine multiqubit entanglement in the vicinity of these states. I also considered entanglement criteria based on the measurement of collective observables. The relation of the entanglement conditions to superradiance was also discussed.

APPENDIX A: PROOF OF INEQUALITY (9)

First, let us fix N_1 and look for the maximum of the left-hand side of inequality (9) as the function of k. (Without loss of generality, we consider $N_1 \le N/2$.) We define

$$g_k := \binom{N_1}{k}\binom{N-N_1}{\frac{N}{2}-k}. \tag{A1}$$

Let us look for the k for which it is maximal. For that we compute the ratio of two consecutive g_k:

$$\frac{g_{k-1}}{g_k} = \frac{k(N/2 - N_1 + k)}{(N_1 - k + 1)(N/2 - k + 1)}. \tag{A2}$$

The right-hand side of Eq. (A2) equals 1 for $k_m = (N_1 + 1)/2$. Thus for $k < k_m$ we know that $g_k/g_{k-1} \ge 1$ while for $k > k_m$ we have $g_k/g_{k-1} \le 1$. Simple calculation shows that the integer value for which g_k is maximal is $k = N_1/2$ for even N_1 and $k = (N_1 \pm 1)/2$ for odd N_1.

Now we know that the maximum of the left-hand side of inequality (9) for a given N_1 is

$$h_{N_1} := \binom{N_1}{\left\lfloor \frac{N_1}{2} \right\rfloor}\binom{N-N_1}{\frac{N}{2} - \left\lfloor \frac{N_1}{2} \right\rfloor}, \tag{A3}$$

where $\lfloor x \rfloor$ denotes the integer part of x. We find that for even N_1,

$$\frac{h_{N_1}}{h_{N_1 \pm 1}} \ge 1. \tag{A4}$$

Hence h_{N_1} must be maximized for some even N_1. Further calculation shows that for even N_1,

$$\frac{h_{N_1}}{h_{N_1-2}} = \frac{N_1 - 1}{N_1}\frac{N - N_1 + 2}{N - N_1 + 1} \le 1. \tag{A5}$$

Hence we know that h_{N_1} is maximized by $N_1 = 2$. Thus we have proved that the left-hand side of inequality (9) is maximized for $N_1 = 2$ and $k = 1$.

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