Generalized spin squeezing criteria: Entanglement detection with collective measurements

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Abstract. What is the relation between spin squeezing and entanglement? To clarify this, we derive the full set of generalized spin squeezing inequalities for the detection of entanglement. These are inequalities for the mean values and variances of the collective angular momentum components. They can be used for the experimental detection of entanglement in a system of spin-$\frac{1}{2}$ particles in which the spins cannot be individually addressed. We also show how to obtain the complete set of entanglement conditions if the mean spin and the spin correlation matrix are known. Finally, we apply our criteria for entanglement detection in spin models, showing that they can be used to detect bound entanglement in the thermal state of these systems.

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INTRODUCTION

Entanglement lies at the heart of many problems in quantum mechanics and has attracted increasing attention in recent years. In physical systems such as ensembles of cold atoms or trapped ions, spin squeezing [1, 2] is one of the most successful approaches for creating large scale quantum entanglement. Since the variance of a spin component is small, spin squeezed states can be used for reducing spectroscopic noise or to improve the accuracy of atomic clocks. Moreover, if an N-qubit state violates the inequality [3]

$$\frac{(\Delta J_x)^2}{\langle J_y\rangle^2 + \langle J_z\rangle^2} \geq \frac{1}{N},$$

where $J_l := \frac{1}{2}\sum_{k=1}^{N} \sigma_l^{(k)}$ for $l = x,y,z$ are the collective angular momentum components and $\sigma_l^{(k)}$ are Pauli matrices, then the state is entangled (i.e., not fully separable), which is necessary for using it in quantum information processing applications.

Recently, several generalized spin squeezing criteria for the detection of entanglement appeared in the literature [4] and have been used experimentally [5]. Such criteria need the measurement of collective spin components and their variances. They have a large practical importance since in many quantum control experiments the spins cannot be individually addressed, and only collective quantities can be measured.

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In this paper we explain the ideas of Ref. [6]. We present the full set of generalized spin squeezing inequalities for many-qubit systems. They detect all entangled states that can be detected knowing the expectation values and variances of spin components. We also show, how this approach can be extended to a set of criteria that detect entanglement based on the correlation matrix. We also discuss that our conditions are able to detect bound entanglement in the ground state of spin models. Surprisingly, these bound entangled states have a positive partial transpose with respect to all bipartitions [7].

GENERALIZED SPIN SQUEEZING CRITERIA

Next, we describe the complete set of generalized spin squeezing entanglement criteria in a system of N spin-$\frac{1}{2}$ particles [6]:

Observation 1. *Let us assume that for a physical system the values of* $\vec{J} := (\langle J_x\rangle, \langle J_y\rangle, \langle J_z\rangle)$ *and* $\vec{K} := (\langle J_x^2\rangle, \langle J_y^2\rangle, \langle J_z^2\rangle)$ *are known. For separable states the following inequalities hold:*

$$\langle J_x^2\rangle + \langle J_y^2\rangle + \langle J_z^2\rangle \leq \tfrac{N(N+2)}{4}, \tag{1a}$$

$$(\Delta J_x)^2 + (\Delta J_y)^2 + (\Delta J_z)^2 \geq \tfrac{N}{2}, \tag{1b}$$

$$\langle J_k^2\rangle + \langle J_l^2\rangle - \tfrac{N}{2} \leq (N-1)(\Delta J_m)^2, \tag{1c}$$

$$(N-1)\left[(\Delta J_k)^2 + (\Delta J_l)^2\right] \geq \langle J_m^2\rangle + \tfrac{N(N-2)}{4}, \tag{1d}$$

where i, j, k take all the possible permutations of x, y, z. The first equation of Eq. (1a) is valid for all quantum states, while if one of the other three inequalities are violated then we know that the state is entangled.

The quantum states fulfilling inequalities Eqs. (1) correspond to a convex object in the six dimensional space of the components of $\vec{J}$ and $\vec{K}$. In order to understand better these inequalities, let us examine what happens if we fix three of these variables. If we fix $\vec{J}$, the points corresponding to separable states in the $(\langle J_x^2\rangle, \langle J_y^2\rangle, \langle J_z^2\rangle)-$space are in a polytope as can be seen in Fig. 1.

It can be proven that for large N the criteria of Observation 1 are complete. Thus, if the inequalities are satisfied, then the first and second moments of J_k do not suffice to prove entanglement. In other words, it is not possible to find criteria detecting more entangled states based on these moments.

CRITERIA WITH THE CORRELATION MATRIX

What happens if we can measure the collective spin in any direction, not only along the x, y and z axes? Knowledge of $\langle J_i\rangle$ and $\langle J_i^2\rangle$ in arbitrary directions is equivalent to the knowledge of the vector $\vec{J}$ and the correlation matrix C defined as

$$C_{kl} := \frac{1}{2}\langle J_k J_l + J_l J_k\rangle \tag{2}$$

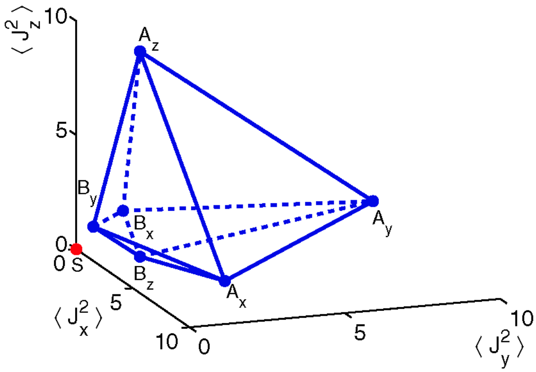

FIGURE 1. The polytope of separable states corresponding to Observation 1 for $N = 6$ and $\vec{J} = 0$. S corresponds to a many body singlet state.

for $k, l = x, y, z$. For convenience, we also define the covariance matrix γ with elements

$$\gamma_{kl} \;:=\; C_{kl} - \langle J_k \rangle \langle J_l \rangle. \tag{3}$$

When changing the coordinate system to x', y', z', vector $\vec{J}$ and the matrices C and γ transform as $\vec{J} \mapsto O\vec{J}$, $C \mapsto OCO^T$ and $\gamma \mapsto O\gamma O^T$ where O is an orthogonal 3×3-matrix. Looking at the inequalities of Observation 1 one finds that the first two inequalities are invariant under a change of the coordinate system. Concerning Eq. (1c), we can reformulate it as

$$\langle J_i^2 \rangle + \langle J_j^2 \rangle + \langle J_k^2 \rangle - \tfrac{N}{2} \leq (N-1)(\Delta J_k)^2 + \langle J_k^2 \rangle. \tag{4}$$

Then, the left hand side is again invariant under rotations, and we find a violation of Eq. (1c) in some direction if the minimal eigenvalue of

$$\mathfrak{X} := (N-1)\gamma + C \tag{5}$$

is smaller than $\mathrm{Tr}(C) - \frac{N}{2}$. Similarly, we find a violation of Eq. (1d) if the largest eigenvalue of $\mathfrak{X}$ exceeds $(N-1)\mathrm{Tr}(\gamma) - N(N-2)/4$.

Observation 2. *We can rewrite our conditions Eqs. (1) in a coordinate system independent way as [6]*

$$\mathrm{Tr}(C) \;\leq\; \tfrac{N(N+2)}{4}, \tag{6a}$$

$$\mathrm{Tr}(\gamma) \;\geq\; \tfrac{N}{2}, \tag{6b}$$

$$\lambda_{\min}(\mathfrak{X}) \;\geq\; \mathrm{Tr}(C) - \tfrac{N}{2}, \tag{6c}$$

$$\lambda_{\max}(\mathfrak{X}) \;\leq\; (N-1)\mathrm{Tr}(\gamma) - \tfrac{N(N-2)}{4}, \tag{6d}$$

where $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ are the smallest and largest eigenvalue of matrix A, respectively. If Eqs. (1) are violated by a quantum state for any choice of coordinate axes x, y, and z then Eqs. (6) are also violated [8].

BOUND ENTANGLEMENT IN SPIN SYSTEMS

One can now ask, what kind of entanglement is detected by our inequalities? Our conditions need only the measurement of two-body correlations and do not have information on higher order correlations. Do they detect only states for which the reduced two-qubit state is entangled? It can be shown that our inequalities can detect entangled states that have a separable two-qubit density matrix, and even states that have a positive partial transpose with respect to all bipartitons. Such entangled states cannot be detected by the Peres-Horodecki criterion [7]. Our criteria can be used to show that such states appear in the thermal state of many spin models, such as for example the Heisenberg and XY models. This way we show that fully PPT bound entanglement appears under natural circumstances.

SUMMARY

We presented the full set of generalized spin squeezing inequalities for multi-qubit systems. These detect all entangled states that can be detected knowing the collective spin components and their variances. We also presented another set of criteria that detect all entanglement that can be detected knowing the expectation values of the collective spin components and the correlation matrix. Our conditions detect multi-qubit bound entanglement in the thermal state of several well-known spin models.

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