Institute of Physics Publishing — Journal of Optics B: Quantum and Semiclassical Optics

J. Opt. B: Quantum Semiclass. Opt. **4** (2002) S213–S217 — PII: S1464-4266(02)30941-8

One-complex-plane representation: a coherent-state description of entanglement and teleportation

J Janszky[1,2], A Gábris[1], M Koniorczyk[1,2], A Vukics[1] and J Asboth[1]

[1] Department of Nonlinear and Quantum Optics, Research Institute for Solid State Physics and Optics, Hungarian Academy of Sciences, PO Box 49, H-1525 Budapest, Hungary
[2] Institute of Physics, University of Pécs, Ifjúság út 6. H-7624 Pécs, Hungary

Received 19 November 2001, in final form 7 February 2002
Published 27 March 2002
Online at stacks.iop.org/JOptB/4/S213

Abstract
It is shown that any state of two modes of the electromagnetic field (or any other bipartite bosonic system) can be expressed as a coherent superposition of conjugate coherent-state pairs. This representation has a deep connection with both the two-mode squeezing operator and a maximally entangled basis. A description of continuous variable quantum teleportation is presented, as an example of the application of this method.

Keywords: Coherent-state representation, two-mode squeezing, entanglement, teleportation

1. Introduction

Multipartite quantum systems described by infinite-dimensional Hilbert spaces (i.e. continuous quantum variables) constitute a possible context for quantum information processing, and are therefore the subject of growing interest in today's research. The most frequently investigated class of quantum states of such systems has been that of Gaussian states, for which several topics such as their separability and distillability properties, [1–5], quantum cloning [6–8], or quantum dense coding [9] have been studied. The basic primitive of quantum communication, teleportation [10] has also been introduced in the continuous variable context [11–13].

Though Gaussian states describe most of the states available experimentally with today's technology, finding alternative representations of *arbitrary* multipartite states is of considerable interest. It is known in quantum optics that the quantum state of a single mode of the electromagnetic field can be generated as superpositions of rather specific sets of appropriately selected coherent states [14–16]. In some cases, even a small finite set of coherent states appears to be sufficient to build up highly nonclassical states of light [17, 18]. But does this way of thinking prove useful in the context of multipartite systems, i.e. multimode light fields? In this paper we present results contributing to the answer to this question.

Our aim is to find a representation of arbitrary states of two-mode light fields (or more generally speaking, bipartite bosonic systems) as superpositions of a certain set of two-mode coherent states, namely pairs of conjugate coherent states:

$$|\alpha\rangle|\alpha^*\rangle \qquad \alpha \in \mathbb{C}. \tag{1}$$

These states have the remarkable property [19] of carrying more information than pair coherent states $|\alpha\rangle|\alpha\rangle$, which also makes them interesting in cloning applications [8]. In addition, they are Gaussian states of a very simple structure.

This paper is organized as follows: in section 2 we provide a brief review of one-dimensional coherent-state representations of single-mode fields. In section 3 the two-mode generalization is provided: we prove that arbitrary two-mode states can be obtained as superpositions of states in equation (1). In section 4 we describe teleportation of continuous variables as an elementary application of our formalism. Section 5 summarizes our results.

2. One-dimensional coherent-state representations revisited

It is well known that the coherent states of a single mode of the electromagnetic field, defined by the eigenvalue equation of the corresponding annihilation operator,

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle \qquad \alpha \in \mathbb{C} \tag{2}$$

form an overcomplete basis. The term 'completeness' stands for the fact that every state of the mode can be expressed as a superposition of coherent states. This expansion, however, is not unique (the reason why this property is labelled *over* complete): there are several proper selections of the weight distributions. A consequence of this ambiguity is that it is possible to select a subset of all coherent states that still forms a complete basis. The mathematical background for these selections was provided by Cahill [20].

The most widespread coherent-state representation involves all coherent states, hence two real parameters (which together correspond to the complex variable α). Using Cahill's theorem, however, it can be shown that a set of coherent states corresponding to a one-dimensional curve on the complex phase space is suitable for representation. This makes it possible to express any state with a superposition of coherent states placed along a one-dimensional curve, instead of taking all possible coherent states of the phase space [15, 16]. A simple example is when the superposition integral goes over either the real or the imaginary axis [14].

A key step towards the application of this kind of representations is to find an orthonormal basis that can be expressed using the one-dimensional subset in argument [15]. On one hand this explicitly proves the completeness of our coherent-state basis, and on the other hand, it provides us with a recipe for finding the appropriate weight function for an arbitrary state. For example, the representation along the real axis is defined with the Hermite polynomials and a Gaussian factor as

$$|h_n\rangle_{\gamma_1} := \mathcal{N}_n(\gamma_1^2)\int_{\mathbb{R}} H_n(\mu x)\exp\left(-\frac{x^2}{\gamma_1^2}\right)|x\rangle\,\mathrm{d}x \qquad \gamma_1\in\mathbb{R}$$

$$\mu = \sqrt{\frac{2(1+\gamma_1^2)}{\gamma_1^2(2+\gamma_1^2)}}. \tag{3}$$

Here $\mathcal{N}_n$ is a normalization factor. With the orthonormal basis at hand, the expansion of every state by means of this restricted set of coherent states becomes rather simple:

$$\begin{aligned}|\psi\rangle &= \sum_{n\in\mathbb{N}}\langle h_n|\psi\rangle|h_n\rangle_{\gamma_1}\\ &= \sum_{n\in\mathbb{N}}\langle h_n|\psi\rangle\mathcal{N}_n(\gamma_1^2)\int_{\mathbb{R}} H_n(\mu x)\exp\left(-\frac{x^2}{\gamma_1^2}\right)|x\rangle\,\mathrm{d}x.\end{aligned} \tag{4}$$

From this formula, a weight distribution

$$F^{\psi}(x) = \sum_{n\in\mathbb{N}}\langle h_n|\psi\rangle\mathcal{N}_n(\gamma_1^2)H_n(\mu x) \tag{5}$$

can be derived that makes the equation

$$|\psi\rangle = \int_{\mathbb{R}} F^{\psi}(x)\exp\left(-\frac{x^2}{\gamma_1^2}\right)|x\rangle\,\mathrm{d}x \tag{6}$$

hold. One has to take care, however, as $F^{\psi}(x)$ exists as an ordinary function only for a limited set of all states, for the rest it can only be interpreted as a distribution. This problem can be overcome by appropriate selection of the coherent-state basis. As a general rule of thumb, a particular one-dimensional coherent-state representation is the most suitable for representing states whose Wigner function 'keeps close' to the one-dimensional curve of the representation. For instance, considering vacuum squeezing with an arbitrary complex squeezing parameter $r_1 = |r_1|\mathrm{e}^{\mathrm{i}\varphi_1}$, the best representation is

$$|r_1\text{squeezed vac.}\rangle = \mathcal{N}(r_1)\int_{\mathbb{R}}\exp\left(-\frac{x^2}{\gamma_1^2}\right)|\mathrm{e}^{\mathrm{i}\varphi_1}x\rangle\,\mathrm{d}x. \tag{7}$$

Here, for convenience, we have introduced $\gamma_1^2 = \mathrm{e}^{2|r_1|} - 1$ which corresponds to the amplitude of the squeezing.

3. Two-mode generalization

Let us now address the question of how the powerful tool of one-dimensional coherent-state representations of section 2 can be generalized to states of two modes. In the former case, the dimensionality of the representation was exactly half of the conventionally available. For two-mode fields the conventional coherent-state representation (i.e. Glauber's analytic representation) involves two complex parameters, hence a double-complex integral of the tensorial product states $|\alpha\rangle|\beta\rangle$. We can reduce the dimensionality of the representation by appropriately choosing a subspace of this phase space. We shall show that a class of subspaces suitable for our purposes may be expressed as

$$(\mathrm{e}^{\mathrm{i}\varrho_a}\alpha, \mathrm{e}^{\mathrm{i}\varrho_b}\alpha^*) \in \mathbb{C}^2. \tag{8}$$

These are planes (two real-dimensional surfaces) embedded in the four real-dimensional $\mathbb{C}^2$ space. Each plane is most conveniently parametrized with a single complex parameter α. We follow the steps of the one-mode case: find an orthogonal basis whose elements can all be expressed as superpositions of the $|\mathrm{e}^{\mathrm{i}\varrho_a}\alpha\rangle|\mathrm{e}^{\mathrm{i}\varrho_b}\alpha^*\rangle$ conjugate coherent-state pairs.

In order to achieve this task, we introduce an orthonormal basis of the Hilbert space of the two modes in argument. This is achieved with the aid of Laguerre-two-dimensional polynomials, derived from Laguerre-two-dimensional functions [21]. We show that these states can all be expressed on our coherent-state basis.

The definition of Laguerre-two-dimensional polynomials reads

$$\begin{aligned}l_{m,n}(z,z^*) &= \frac{1}{\sqrt{\pi}}\frac{1}{\sqrt{m!n!}}\\ &\times\sum_{j=0}^{\min\{m,n\}}\frac{m!n!}{j!(m-j)!(n-j)!}(-1)^j z^{*m-j}z^{n-j}.\end{aligned} \tag{9}$$

This special set of polynomials on the complex plane is complete with the weight function e^{-zz^*}:

$$\sum_{m,n\in\mathbb{N}}\mathrm{e}^{-zz^*}l_{m,n}(z,z^*)l_{m,n}^*(w,w^*) = \delta(z-w, z^*-w^*) \tag{10}$$

and orthonormal

$$\int_{\mathbb{C}}\mathrm{e}^{-zz^*}l_{k,l}^*(z,z^*)l_{m,n}(z,z^*)\,\mathrm{d}^2z = \delta_{k,m}\delta_{l,n}. \tag{11}$$

With the aid of these polynomials, we define the set of states (in analogy with those defined with Hermite polynomials in the one-mode case):

$$\begin{aligned}|l_{m,n}\rangle_{\gamma}^{\{\varrho\}} &= \mathcal{N}_{m,n}(\gamma)\int_{\mathbb{C}}\\ &\times\exp\left(-\frac{|\alpha|^2}{\gamma^2}\right)l_{m,n}(\mu\alpha,\mu\alpha^*)|\mathrm{e}^{\mathrm{i}\varrho_a}\alpha\rangle|\mathrm{e}^{\mathrm{i}\varrho_b}\alpha^*\rangle\,\mathrm{d}^2\alpha\end{aligned} \tag{12}$$

where

$$\mathcal{N}_{m,n}(\gamma) = \frac{\sqrt{1+2\gamma^2}}{\gamma^2\sqrt{\pi}}\left(\frac{1+\gamma^2}{\gamma^2}\right)^{\frac{m+n}{2}}$$
$$\mu = \sqrt{\frac{1+2\gamma^2}{\gamma^2(\gamma^2+1)}}. \tag{13}$$

Let $\mathcal{L} := \{|l_{m,n}\rangle_\gamma^{\{\varrho\}} | m, n \in \mathbb{N}\}$. We shall now prove that $\mathcal{L}$ is a complete orthonormal basis of the Hilbert space. One way of doing so is to present a unitary mapping from a well known complete orthonormal basis to $\mathcal{L}$. In what follows, we shall see that a suitable choice can be the number-state basis, which in turn implies that the unitary mapping is two-mode squeezing.

In the general case, the two-mode squeezing operator is written

$$\hat{S}^{(2)}(r) = \mathrm{e}^{-r\hat{a}^\dagger\hat{b}^\dagger + r^*\hat{a}\hat{b}} \tag{14}$$

where $r = |r|\mathrm{e}^{\mathrm{i}\varphi} \in \mathbb{C}$ is the squeezing parameter. It is possible to express the action of $\hat{S}^{(2)}(r)$ on the ladder operators as

$$\hat{a}' = u\hat{a} + v\hat{b}^\dagger$$
$$\hat{b}' = u\hat{b} + v\hat{a}^\dagger \tag{15}$$

where we have introduced $u = \cosh|r|$ and $v = \mathrm{e}^{\mathrm{i}\varphi}\sinh|r|$. These satisfy the following relations:

$$|u|^2 - |v|^2 = 1$$
$$u > 0 \tag{16}$$

which together imply the correct commutation relations for the $\hat{a}'$ and $\hat{b}'$ ladder operators, together with the inequality $u > |v|$.

Let us compare the two-mode squeezed vacuum, $\hat{S}^{(2)}(r)|0,0\rangle$ to $|l_{0,0}\rangle_\gamma^{\{\varrho\}}$. This comparison can most conveniently be done with characteristic functions: for the two-mode squeezed vacuum we have

$$\chi_S(\eta,\xi) = \langle 0,0|\hat{S}^{(2)\dagger}\mathrm{e}^{\eta\hat{a}^\dagger}\mathrm{e}^{\eta^*\hat{a}}\mathrm{e}^{\xi\hat{b}^\dagger}\mathrm{e}^{\xi^*\hat{b}}\hat{S}^{(2)}(r)|0,0\rangle \tag{17}$$

whereas for $|l_{0,0}\rangle_\gamma^{\{\varrho\}}$ it is

$$\chi_L(\eta,\xi) = \langle l_{0,0}|_\gamma \mathrm{e}^{\eta\hat{a}^\dagger}\mathrm{e}^{\eta^*\hat{a}}\mathrm{e}^{\xi\hat{b}^\dagger}\mathrm{e}^{\xi^*\hat{b}}|l_{0,0}\rangle_\gamma^{\{\varrho\}}. \tag{18}$$

We evaluate equation (17) by performing the substitutions described in equation (15), and for equation (18) we make use of the eigenvalue equation mentioned in equation (2). After some algebra, we notice that χ_S and χ_L become equal if we take $\gamma = \sqrt{|v|/(u-|v|)}$ and $\varrho_a + \varrho_b = \varphi + \pi$. For the states under discussion, this means that

$$\hat{S}^{(2)}(r)|0,0\rangle = |l_{0,0}\rangle_\gamma^{\{\varrho\}}. \tag{19}$$

Considering an arbitrary Fock state $|m,n\rangle$ the transformation is

$$\hat{S}^{(2)}(r)|m,n\rangle = \hat{S}^{(2)}(r)\frac{\hat{a}^{\dagger m}}{\sqrt{m!}}\frac{\hat{b}^{\dagger n}}{\sqrt{n!}}|0,0\rangle$$
$$= \frac{1}{\sqrt{m!n!}}(u\hat{a}^\dagger + v^*\hat{b})^m(u\hat{b}^\dagger + v^*\hat{a})^n\hat{S}^{(2)}(r)|0,0\rangle$$
$$= \frac{1}{\sqrt{m!n!}}(u\hat{a}^\dagger + v^*\hat{b})^m(u\hat{b}^\dagger + v^*\hat{a})^n|l_{0,0}\rangle_\gamma^{\{\varrho\}}. \tag{20}$$

We evaluate this expression using the identities

$$\int_{\mathbb{C}}\exp\left(-\frac{|\alpha|^2}{\gamma^2}\right)(\hat{a}^\dagger)^n|\mathrm{e}^{\mathrm{i}\varrho_a}\alpha\rangle|\mathrm{e}^{\mathrm{i}\varrho_b}\alpha^*\rangle\,\mathrm{d}^2\alpha$$
$$= \int_{\mathbb{C}}\exp\left(-\frac{|\alpha|^2}{\gamma^2}\right)\left(-\frac{v}{u}\alpha^*\right)^n|\mathrm{e}^{\mathrm{i}\varrho_a}\alpha\rangle|\mathrm{e}^{\mathrm{i}\varrho_b}\alpha^*\rangle\,\mathrm{d}^2\alpha$$
$$\int_{\mathbb{C}}\exp\left(-\frac{|\alpha|^2}{\gamma^2}\right)(\hat{b}^\dagger)^n|\mathrm{e}^{\mathrm{i}\varrho_a}\alpha\rangle|\mathrm{e}^{\mathrm{i}\varrho_b}\alpha^*\rangle\,\mathrm{d}^2\alpha$$
$$= \int_{\mathbb{C}}\exp\left(-\frac{|\alpha|^2}{\gamma^2}\right)\left(-\frac{v}{u}\alpha\right)^n|\mathrm{e}^{\mathrm{i}\varrho_a}\alpha\rangle|\mathrm{e}^{\mathrm{i}\varrho_b}\alpha^*\rangle\,\mathrm{d}^2\alpha \tag{21}$$

involving the creation operators. These identities may look quite surprising at first sight, as it looks as though the coherent states were eigenstates of the creation operators also. However, one has to notice that this behaviour concerning the creation operators may well exist only for this special integral formula. By repeated applications of equations (2) and (21) and the commutation relations for the ladder operators, we see that

$$\hat{S}^{(2)}(r)|m,n\rangle = |l_{m,n}\rangle_\gamma^{\{\varrho\}} \tag{22}$$

if we omit the global phase factors. Hence, we have proven that $\mathcal{L}$ is a complete (orthogonal) basis for any value of γ and $\varrho_a+\varrho_b$.

By looking at the formulae in equation (12), we notice that for two squeezing operators with the same amplitude but different phase ($\mathrm{e}^{\mathrm{i}\varphi}$) the $|l_{m,n}\rangle_\gamma^{\{\varrho\}}$ states differ also only in the phase factors $\mathrm{e}^{\mathrm{i}\varrho_a}$ and $\mathrm{e}^{\mathrm{i}\varrho_b}$. This difference corresponds to a time-evolution-like unitary transformation. The similar behaviour of the one-dimensional representation was mentioned earlier in this paper. When investigating the nature of this representation, this property permits us to consider only the squeezing leading to the most simple formulae. Namely the case when when r is real and negative, hence $\mathrm{e}^{\mathrm{i}\varphi} = -1$. This also makes possible the $\varrho_a = \varrho_b = 0$ selection, leading to the integral formula

$$|l_{m,n}\rangle_\gamma = \mathcal{N}_{m,n}(\gamma)\int_{\mathbb{C}}\exp\left(-\frac{|\alpha|^2}{\gamma^2}\right)l_{m,n}(\mu\alpha,\mu\alpha^*)|\alpha\rangle|\alpha^*\rangle\,\mathrm{d}^2\alpha. \tag{23}$$

Investigating the nature of this representation, an obvious question is to ask, what is the expression of the identity operator in the $|\alpha\rangle|\alpha^*\rangle$ basis? Since equation (22) holds, it can be acquired by the sum

$$\hat{I}^{(2)} = \sum_{m,n\in\mathbb{N}}|l_{m,n}\rangle_\gamma\langle l_{m,n}|_\gamma. \tag{24}$$

It can be shown, however, that both integrals persist in this formula, thus even the reduced-dimensionality coherent-state expansions of the two-mode identity operator involve a double-complex integral.

The $\mathcal{L}$ basis has yet another interesting property, which is best revealed when it is expressed in the number-state basis. To keep the formulae more compact, we treat the $m \geqslant n$ and $m < n$ cases separately. Indeed, the calculations need to be done for one case only, as a result of the symmetry for m and n. After some algebra, for $m \geqslant n$ we obtain

$$|l_{m,n}\rangle_\gamma = \mathcal{N}_{m,n}(\gamma)\sum_{l\in\mathbb{N}}C^l_{m,n}|m-n+l\rangle|l\rangle \tag{25}$$

where

$$C^l_{m,n} = \sum_{j=0}^{n} \frac{1}{\sqrt{m!n!k!l!}} \frac{m!n!}{j!(m-j)!} (-u|v|)^j \times \frac{(n+k-j)!}{(n-j)!} \left(\frac{u}{|v|}\right)^{n+k-j} \quad (26)$$

and

$$k = m - n + l. \quad (27)$$

Since (25) is the Schmidt decomposition of $|l_{m,n}\rangle_\gamma$, its entanglement can be expressed through $N_{m,n}(\gamma)$ and $C^l_{m,n}$ as

$$E(|l_{m,n}\rangle_\gamma) = -\sum_{l\in\mathbb{N}} \mathcal{N}_{m,n}(\gamma) C^l_{m,n} \log(\mathcal{N}_{m,n}(\gamma) C^l_{m,n}). \quad (28)$$

By the nature of these coefficients, $E(|l_{m,n}\rangle_\gamma) > 0$ holds for all $m, n \in \mathbb{N}$ pairs, and also

$$\lim_{\gamma\to\infty} E(|l_{m,n}\rangle_\gamma) = 1 \quad (29)$$

corresponding to maximal entanglement. Thus $\mathcal{L}$ is a good approximation of a Bell-basis in this infinite-dimensional Hilbert space.

Notice the remarkable fact that two-mode squeezing, which has a decisive role in entanglement of Gaussian states, provides the transformation from the Fock-state basis to a basis that can be expressed as the superposition of the conjugate-phase coherent-state pairs, which approximate maximally entangled states.

4. Continuous variable quantum teleportation in coherent-state basis

In this section we shall consider the formulation of continuous variable teleportation in a coherent-state basis. This can be regarded as a simple application of coherent-state representations introduced in section 3. Here we adopt a less formal argument: we follow the steps of the protocol, and construct the one-complex-plane representation of two-mode states in argument via simple physical considerations.

The scheme proposed by Kimble *et al* [12] for teleporting continuous quantum variables involves a sender named Alice and a recipient named Bob. At the beginning of the process, Alice possesses the unknown state in mode 1 and a part of an entangled (EPR) state in mode 2, with the other part in mode 3 being at Bob. Then she performs a joint measurement on modes 1 and 2. This measurement must project onto a maximally entangled basis, the so-called quadrature Bell states in our case. To reconstruct the original state, Bob has to carry out one of the specific unitary transformations on his part of the EPR state in mode 3. Since this transformation must be appropriate for the actual projection made, the measurement outcome has to be sent to Bob via a classical channel.

The EPR state used as entangled resource in the protocol is the two-mode squeezed vacuum state. According to equation (19), this may be written

$$|\Psi^\gamma_{EPR}\rangle_{23} = |l_{0,0}\rangle_{\gamma/\sqrt{2}} = \mathcal{N}_{0,0} \int_{\mathbb{C}} \exp\left(-\frac{2|\alpha|^2}{\gamma^2}\right) |\alpha\rangle|\alpha^*\rangle \, d^2\alpha. \quad (30)$$

Notice that the usage of $\gamma/\sqrt{2}$ instead of γ does not change the fact that as $\gamma \to \infty$, we approach the infinite squeezing limit, which produces a maximally entangled state according to equation (29).

Let us turn to the quadrature Bell states. To realize this measurement, modes 1 and 2 are directed to the input ports of a 50–50% beamsplitter. Then the $\hat{X}$ and $\hat{P}$ quadrature operators are measured on each of the respective output modes. With ideal homodyne detectors this can be interpreted as a projection onto the quadrature eigenstates. To express these eigenstates in a two-dimensional coherent-state basis, consider first the eigenstates for $X = 0$ and $P = 0$. They correspond to the infinitely squeezed vacua

$$\begin{aligned} \|X = 0\rangle &= \lim_{\gamma_1\to\infty} \int_{\mathbb{R}} \exp\left(-\frac{y^2}{\gamma_1^2}\right) |iy\rangle \, dy = \int_{\mathbb{R}} |iy\rangle \, dy \\ \|P = 0\rangle &= \lim_{\gamma_1\to\infty} \int_{\mathbb{R}} \exp\left(-\frac{x^2}{\gamma_1^2}\right) |x\rangle \, dx = \int_{\mathbb{R}} |x\rangle \, dx \end{aligned} \quad (31)$$

derived from equation (7) with $e^{i\varphi_1} = i$ and $e^{i\varphi_1} = 1$, respectively. We denote these states by double-stroke kets. These states are obviously unnormalizable, which is the usual property of ‘position’ and ‘momentum’ eigenstates. Normalization factors can be omitted for this reason. The rest of the quadrature eigenstates can be obtained by applying the proper coherent displacement operation

$$\begin{aligned} \|X\rangle &= \hat{D}(X)\|X = 0\rangle = \int_{\mathbb{R}} e^{-iXy} |X + iy\rangle \, dy \\ \|P\rangle &= \hat{D}(iP)\|P = 0\rangle = \int_{\mathbb{R}} e^{ixP} |x + iP\rangle \, dx. \end{aligned} \quad (32)$$

Here $\hat{D}(\alpha) = e^{\alpha\hat{a}^\dagger - \alpha^*\hat{a}}$ denotes the Glauber displacement operator. Ideal homodyne detectors realize a projection onto these $\|X\rangle_1\|P\rangle_2$ states. A beamsplitter is used to turn this projection into a Bell state measurement. The inverse beamsplitter transform can be easily calculated in the coherent-state representation, since the coherent states interfere as classical amplitudes:

$$|\alpha\rangle_{1'}|\beta\rangle_{2'} \to |(\alpha + \beta)/\sqrt{2}\rangle_1 |(\beta - \alpha)/\sqrt{2}\rangle_2. \quad (33)$$

Because of the linearity of integration, this formula can be instantly applied to $\|X\rangle_1\|P\rangle_2$, and we get

$$|\Psi^{(A)}_{Bell}\rangle = \int_{\mathbb{C}} e^{A\alpha^* - A^*\alpha} |\alpha + A\rangle_1 |\alpha^* - A^*\rangle_2 \, d^2\alpha \quad (34)$$

for the Bell states, where we have introduced the new complex variables

$$\alpha := \frac{x + iy}{\sqrt{2}} \qquad \text{and} \qquad A := \frac{X + iP}{\sqrt{2}}. \quad (35)$$

It is worth mentioning here that the resulting formula for the $|\Psi^{(A)}_{Bell}\rangle$ Bell states suggests a somewhat different approach to representation on one complex plane than the one mentioned in section 3. There the unitary mapping was the two-mode squeezing operator, and the corresponding basis the number states. Here, however, a different basis, the quadrature eigenstates were transformed by a beamsplitter to form a new, entangled basis.

So far we have prepared all the tools needed to describe the projective measurement on the unknown state and Alice's part of the EPR pair. Let the unknown state be written in the Glauber analytic representation as

$$|\Psi_{in}\rangle_1 = \int_{\mathbb{C}} \mathrm{e}^{-\frac{|\alpha|^2}{2}} f(\alpha^*)|\alpha\rangle_1 \, \mathrm{d}^2\alpha. \tag{36}$$

The joint state of the three-mode system is

$$|\Psi_{joint}\rangle_{123} = |\Psi_{in}\rangle_1 |\Psi^{\gamma}_{EPR}\rangle_{23}. \tag{37}$$

If the measurement outcome corresponds to A, this state has to be projected onto $|\Psi^{(A)}_{Bell}\rangle$ in modes 1 and 2. To obtain the state of mode 3 after the measurement, the partial inner product,

$$|\Psi_t\rangle_3 = {}_{12}\langle\Psi^{(A)}_{Bell}|\Psi_{joint}\rangle_{123} \tag{38}$$

has to be calculated. Considering (34) and (36), this expression yields a double complex integral. One of these integrals can be eliminated (for details, see [22]). We obtain the state

$$|\Psi_t\rangle_3 = \int_{\mathbb{C}} \exp\left(-\frac{2|\alpha - 2A|^2}{\gamma^2}\right) \hat{D}(-2A) \times \exp\left(-\frac{|\alpha|^2}{2}\right) f(\alpha^*)|\alpha\rangle_3 \, \mathrm{d}^2\alpha \tag{39}$$

for mode 3 after the measurement. In the ideal case, when the EPR pair is maximally entangled (infinite squeezing limit, $\gamma \rightarrow \infty$), the Gaussian factor $\exp(-(2|\alpha - 2A|^2)/(\gamma^2))$ becomes equal to one. The original state $|\Psi_{in}\rangle_1$ can then be restored by Bob using the unitary operation $\hat{D}(2A)$, a displacement. This is in agreement with other descriptions of quantum teleportation [12, 23, 24].

In a more realistic case, however, γ has a finite value. In that case the original state is not reproduced exactly by the same operation, since a Gaussian smoothing factor appears in the integral

$$|\Psi_f\rangle = \int_{\mathbb{C}} \exp\left(-\frac{2|\alpha - 2A|^2}{\gamma^2}\right) \exp\left(-\frac{|\alpha|^2}{2}\right) f(\alpha^*)|\alpha\rangle_3 \, \mathrm{d}^2\alpha. \tag{40}$$

This result is not surprising [25], as it is a consequence of the finite number of photons used to prepare the EPR state.

5. Conclusion

We have shown that an *arbitrary* state of two modes of the electromagnetic field (or any other bipartite bosonic system) can be expressed as a coherent superposition of conjugate coherent-state pairs. This is a representation of one complex dimension instead of two, therefore we call it one-complex-plane representation. The two-mode squeezing operator plays an important role in the establishment of this result, namely it transforms the Fock-state basis to an orthonormal basis with elements easily expressible as a form of the superpositions in the argument. This latter basis approximates a maximally entangled basis of the two-mode Hilbert space. We have presented the description of continuous variable quantum teleportation, an elementary example of the application of this representation.

Regarding the known interesting information-theoretical properties of the conjugate coherent-state pairs, this representation may prove useful in quantum information processing research, for instance for quantum cloning of arbitrary states. Coherent states are simple prototypes of Gaussian states. Using them as a basis of representation may help to extend our present knowledge on Gaussian states to more wider classes of multipartite non-classical states.

Acknowledgments

The authors would like to thank Z Kis for his valuable remarks on this topic. This work was supported by the Research Fund of Hungary (OTKA) under contract no T034484.

References

[1] Parker S, Bose S and Plenio M B 2000 *Phys. Rev.* A **61** 032305

[2] Duan L-M, Giedke G, Cirac J I and Zoller P 2000 *Phys. Rev.* A **62** 032304 (quant-ph/0007061)

[3] Giedke G, Kraus B, Duan L-M, Zoller P, Cirac J I and Lewenstein M 2001 *Fortschr. Phys.* **49** 973
Giedke G, Duan L-M, Zoller P and Cirac J I 2001 *Preprint* quant-ph/0104072

[4] Giedke G, Krauss B, Cirac J I and Lewenstein M 2001 *Phys. Rev.* A **64** 052303

[5] Duan L-M, Giedke G, Cirac J I and Zoller P 2000 *Phys. Rev. Lett.* **84** 4002

[6] Braunstein S L, Bužek V and Hillery M 2001 *Phys. Rev.* A **63** 052313

[7] Fiurášek J 2001 *Phys. Rev. Lett.* **86** 4942

[8] Cerf N J and Iblisdir S 2001 *Phys. Rev. Lett.* **87** 247903 (quant-ph/0102077)

[9] Braunstein S L and Kimble H J 2000 *Phys. Rev.* A **61** 042302

[10] Bennett C H, Brassard G, Crépeau C, Jozsa R, Peres A and Wootters W K 1993 *Phys. Rev. Lett.* **70** 1895

[11] Vaidman L 1994 *Phys. Rev.* A **49** 1473

[12] Braunstein S L and Kimble H J 1998 *Phys. Rev. Lett.* **80** 869

[13] Furusawa A, Sørensen J L, Braunstein S L, Fuchs C A, Kimble H J and Polzik E S 1998 *Science* **282** 706

[14] Janszky J and Vinogradov A V 1990 *Phys. Rev. Lett.* **64** 2771

[15] Adam P, Földesi I and Janszky J 1994 *Phys. Rev.* A **49** 1281

[16] Janszky J, Domokos P and Adam P 1993 *Phys. Rev.* A **48** 2213

[17] Janszky J, Domokos P, Szabó S and Adam P 1995 *Phys. Rev.* A **51** 4191

[18] Szabo S, Adam P, Janszky J and Domokos P 1996 *Phys. Rev.* A **53** 2698

[19] Cerf N J and Iblisdir S 2001 *Phys. Rev.* A **64** 032307 (quant-ph/0012020)

[20] Cahill K E 1965 *Phys. Rev.* **138** B1566

[21] Wünsche A 1998 *J. Phys. A: Math. Gen.* **31** 8267

[22] Janszky J, Koniorczyk M and Gábris A 2001 *Phys. Rev.* A **64** 034302

[23] Yu S and Sun C-P 2000 *Phys. Rev.* A **61** 022310

[24] Koniorczyk M, Bužek V and Janszky J 2001 *Phys. Rev.* A **64** 034301

[25] Hofmann H F, Ide T, Kobayashi T and Furusawa A 2000 *Phys. Rev.* A **62** 062304