Chaos in the conditional dynamics of two qubits purification protocol

S. Vymětal*, T. Kiss†, I. Jex*, G. Alber**, A. Gábris*,† and T. Langrová*

*Department of Physics, FJFI ČVUT, Břehová 7, 115 19 Praha 1 - Staré Město, Czech Republic
†Research Institute for Solid State Physics and Optics, H-1525 Budapest, P.O. Box 49, Hungary
**Institut für Angewandte Physik, Tech. Universität Darmstadt, D-64289 Darmstadt, Germany

Abstract. The presence of complex chaos in iterative applications of selective dynamics on quantum systems is a novel form of quantum chaos with true sensitivity to initial conditions. Techniques for the study of pure states are extended to the two-qubit case[1].

Keywords: complex chaos, quantum chaos, purification protocol
PACS: 3.67.Lx, 05.45.Mt, 42.50.Lc, 89.70.-a

INTRODUCTION

Purification protocols which decrease the mixedness of an ensemble of entangled states are important processes in the theory of quantum information since entanglement plays a key role for the construction of quantum computers, quantum teleportation etc. Conditional dynamics are at the heart of the purification protocol described in [1], introducing nonlinearity in the dynamics. As a consequence, after only a few iterations the dynamics become complicated with high sensitivity to initial conditions. We have proven [2, 3, 4] that the dynamics of the purification protocol is truly chaotic even for the simplest case of an ensemble of pure qubits and the speed of divergency is exponential. Here the dynamics is governed by a map over complex numbers, therefore we may call it complex chaos. The presence of complex chaos can be viewed in two ways - positive and negative. On one hand, it can completely destroy the purification procedure: a small change in the setting of a protocol or a small disturbance on an ensemble of purified states can lead the process to chaos. But on the other hand, such sensitivity of the protocol might be used with advantages to distinguish very close quantum states therefore the protocol could be used as a kind of quantum microscope [5].

PURIFICATION PROTOCOL

The key point of the purification protocol proposed in [1] is the nonlinearity in the iteration of the considered quantum system. To achieve it, one needs to have an ensemble of identically prepared systems which, in the simplest case, are divided into pairs of control-target systems. In one iteration step a generalized *XOR* gate together with a pro-

[1] Send correspondence to S. Vymětal, e-mail: stanislav.vymetal@gmail.com

CP1076, *Let's Face Chaos Through Nonlinear Dynamics: 7th International Summer School and Conference*
edited M. Robnik and V. Romanovski

jective measurement of the target system is applied on each such pair. By choosing the right projector, the elements of the density matrix of the control system can be (for example) "squared", which in effect significantly decreases small elements. Moreover, each iteration step is augmented by an additional abstract rotation of the control system (the most general unitary transformation on the control system) to support the purification of the system with desired symmetry. We will present here this construction for qubits systems.

Let T form the nonlinear part of the map on the control-target pair i.e.

$$T : \mathscr{B}(\mathscr{H}) \otimes \mathscr{B}(\mathscr{H}) \mapsto \mathscr{B}(\mathscr{H}) \otimes \mathscr{B}(\mathscr{H}),$$

where $\mathscr{H} = \mathscr{H}(2)^{\otimes M}$, and let us define T as

$$T\left(\sigma^c, \sigma^t\right) = \frac{A\,(\sigma^c \otimes \sigma^t)\,A^\dagger}{Tr[A\,(\sigma^c \otimes \sigma^t)\,A^\dagger]}, \tag{1}$$

where the operator A is defined as

$$A = (\mathbf{1_c} \otimes \mathbf{P}_t)\,\Pi_{j=1}^{M} XOR_{ct}^{j},\ \mathbf{P}_t = |\mathbf{p}\rangle_{tt}\,\langle\mathbf{p}| \tag{2}$$

with XOR_{ct}^{j} which represents here a XOR gate acting onto the j-th qubit of the control and target system i.e.

$$XOR_{ct}^{j}\,|\mathbf{k}\rangle_c|\mathbf{l}\rangle_t = |\mathbf{k}\rangle_c|\tilde{\mathbf{l}}\rangle_t, \tag{3}$$

where $|\tilde{\mathbf{l}}\rangle = |l_1, \ldots, l_{j-1}, \left(k_j - l_j\right) \bmod 2, l_{j+1}, \ldots, l_M\rangle$.

Such map is generally nonlinear with respect to the action on the elements of the density matrix σ^c. For example if $\mathbf{P}_t = |\mathbf{0}\rangle\langle\mathbf{0}|$, then σ^c is mapped onto $\overline{\sigma}^c$, where $\overline{\sigma}^c_{ij} = 1/N\,(\sigma^c_{ij})^2$ and the normalization $N = \sum_{ii}\sigma^c_{ii}$. Various $\mathbf{P}_t$ form completely different protocols.

To control the map, let us add a corrective unitary transformation

$$R : \overline{\sigma}^c \mapsto \sigma^c_{out} = U_1 \otimes \ldots \otimes U_M \overline{\sigma}^c U_M^\dagger \otimes \ldots \otimes U_1^\dagger, \tag{4}$$

where the general form of the particle rotation is

$$U_i = \begin{pmatrix} \cos(x_i) & \sin(x_i)e^{\imath\phi_i} \\ -\sin(x_i)e^{-\imath\phi_i} & \cos(x_i) \end{pmatrix}. \tag{5}$$

One iterative step F of the purification protocol is composed of the actions of T and R i.e.

$$F \equiv F|_c : \sigma_c \mapsto (R \circ T|_c)\sigma_c. \tag{6}$$

ONE QUBIT MAP

As shown in [2], the dynamics of the presented purification protocol displays highly nontrivial dynamics with an extreme sensitivity to initial conditions, which leads to the presence of chaos.

Let us sum up the simplest case of a purification protocol for the case of pairs of qubits. The nonlinear part of the protocol is strongly influenced by the choice of the projection $\mathbf{P}_t = |p\rangle\langle p|$ where $p = \{0,1\}$. If $p = 0$, the nonlinear part consist of "squaring" i.e.

$$\sigma^c = \begin{pmatrix} \sigma_{00} & \sigma_{01} \\ \sigma_{10} & \sigma_{11} \end{pmatrix} \mapsto \overline{\sigma}^c = \begin{pmatrix} \sigma_{00}^2 & \sigma_{01}^2 \\ \sigma_{10}^2 & \sigma_{11}^2 \end{pmatrix}, \tag{7}$$

which maps a pure state on a pure state and $|0\rangle$, $|1\rangle$, $1/\sqrt{2}(|0\rangle + |1\rangle)$ are fixed points. If $p = 1$, the nonlinear part can act as a filter

$$\sigma^c \mapsto \overline{\sigma}^c = \begin{pmatrix} 1 & \xi \\ \xi & 1 \end{pmatrix},\ \xi = \frac{\sigma_{01}\sigma_{10}}{\sigma_{00}\sigma_{11}}, \sigma_{00} \wedge \sigma_{11} \neq 0. \tag{8}$$

The pure states $|0\rangle$ and $|1\rangle$ are annihilated and the remaining pure states are mapped onto the pure state $1/\sqrt{2}(|0\rangle + |1\rangle)$. If the purification protocol consists only of a chain of steps with the nonlinear part $T_{p=0}$, the perturbation in p has severe impact mainly if we want to purify towards the states $|0\rangle$, $|1\rangle$. $T_{p=1}$ itself is so strong that the chain identity satisfy

$$T_{p=0}T_{p=1}T_{p=0} \equiv T_{p=1}T_{p=1}T_{p=0}. \tag{9}$$

Let us study the dynamics of the purification protocols in the sense of [2]. Pure states are mapped onto pure states or annihilated. Generally a pure state is described via two complex parameters restricted by a normalization. To study the dynamics of a system, it is favorable to decrease the number of parameters to a minimum. For this purpose, it is possible to use the fact that pure states are physically equivalent up to a global phase. Then all physically equivalent states $\{|\psi\rangle = \alpha|0\rangle + \beta|1\rangle\}$ are equal to

$$|\psi(z)\rangle = N(z)(z|0\rangle + |1\rangle), \tag{10}$$

where $N(z)$ is normalization. In this notation, $z = \infty$ represents $|\psi\rangle \equiv |0\rangle$ and $z = 0$ represents $|\psi\rangle \equiv |1\rangle$.

One step of the purification protocol maps $|\psi(z)\rangle$ onto $|\psi(\mathscr{F}(z))\rangle$ where $\mathscr{F}_p : \hat{\mathbb{C}} \mapsto \hat{\mathbb{C}}$ and $\hat{\mathbb{C}} = \mathbb{C} \cup \infty$. Depending on the value of p, $\mathscr{F}$ can have one the forms

$$\mathscr{F}_{p=0}(z) = \frac{z^2 + a}{1 - a^* z^2} \tag{11}$$

$$\mathscr{F}_{p=1}(z) = \frac{1 + a}{1 - a^*} = \text{const.} \equiv C_a \tag{12}$$

where the parameter $a = \tan(x)\, e^{\imath\phi}$.

One can see that the dynamics of $\mathscr{F}_{p=1}$ is simple. The action on a state with $z \in \{0, \infty\}$ annihilates the control system. The initial state $|\psi(C_a)\rangle$ is a stable fixed point of the n-th order for $a \neq \{-1, \imath\}$ and $\mathscr{F}_{p=1}$ purifies to $|\psi(C_a)\rangle$. For $a \in \{-1, \imath\}$ no fixed point exists and $\mathscr{F}^{\circ 2}{}_{p=1}$ behaves as a filter of pure states (mixed states are not annihilated in general).

As was shown in [2], the dynamics of $\mathscr{F}_{p=0}$ is more interesting. $\mathscr{F}_{p=0}$ is an endomorphism of $\hat{\mathbb{C}}$, according to [6], such map is chaotic on its Julia set.

It is easy to study the protocol for $a = 0$ when the protocol is in fact without the additional unitary transformation. Then $\mathscr{F}_{p=0} = z^2$, the protocol $\mathscr{F}^{\circ n}_{p=0}(z)$ has $2n$ fixed points $z \in \{0,\ e^{2k\pi/2n-1}\}$, $k \in \{0, 1, .., 2(n-1)\}$, but only the fixed point for $z = 0$ is stable, the remaining ones are unstable. The Julia set is the circle $K(0,1) \in \hat{\mathbb{C}}$, the states inside and outside the Julia set converge to the states $|1\rangle$ and $|0\rangle$ respectively.

If $a \neq 0$ then $\mathscr{F}_{p=0}$ is a rational map of order 2. According to [6] and [7], $\mathscr{F}_{p=0}$ has at most three fixed points with multiplicity, its Julia set is not empty and the map has cycles of periodic points with every prime except for the three. The map has two critical points: $z_{c_1} = 0 \leftrightarrow |1\rangle$, $z_{c_2} = \infty \leftrightarrow |0\rangle$. $\mathscr{F}_{p=0}$ has no indifferent cycles and at most six repelling cycles. The fact that the Julia set is not empty can be seen for the case when $a = 1$. In this case, z_{2c} is part of the attractive cycle $C = (-1, \infty)$ and z_{1c} follows the orbit $0 \mapsto 1 \mapsto \infty$ which leads to the cycle C. Then $\mathscr{F}_{p=0}$ is hyperbolic and consequently the Julia set is not empty.

TWO QUBITS MAP

Let us study a purification protocol where each control and target system consists of two qubits, i.e. where $\mathscr{H} = \mathscr{H}(2) \otimes \mathscr{H}(2)$. The nonlinear part is influenced by the projection $\mathbf{P}_t$ which has now four possibilities $\mathbf{P}_t = |\mathbf{p}\rangle\langle\mathbf{p}|$, $\mathbf{p} \equiv (i,j)$, $i,j \in \{0,1\}$. The additional rotation generally consists of the action $U_1 \otimes U_2$. The purification protocol can be now used for purification towards entangled states (Bell states) $|\Psi_{1,2}\rangle = 1/\sqrt{2}(|01\rangle \pm |10\rangle)$, $|\Psi_{3,4}\rangle = 1/\sqrt{2}(|00\rangle \pm |11\rangle)$. Generally, the purification protocol does not preserve pure states.

Let us study a protocol which preserves pure states i.e. a protocol where $\mathbf{p} = (\mathbf{0}, \mathbf{0})$ and $U_1 = U_2$. Each pure state can be represented now as $|\psi\rangle = \alpha_1|e_1\rangle + \alpha_2|e_2\rangle + \alpha_3|e_3\rangle + \beta|e_4\rangle$, using some orthogonal basis. If one represents the pure state $|\psi\rangle$ via three independent complex variables $\{z_i = \alpha_i/\beta\}$ (similar to the one qubit map approach (10)), difficulties with singularities in z_i arise. Namely, for every state $|\psi\rangle \in \text{span}\{|e_1\rangle, |e_2\rangle, |e_3\rangle\}$ we have at least one $z_i = \infty$. If two of the $z_i = \infty$ the representation becomes ambiguous. Nevertheless, the representation can be used for the study of stable fixed points, since one can always choose the basis states $|e_i\rangle$, such that β does not vanish in the neighbourhood of the fixed point. With respect to the mentioned limitations let us illustrate the complexity of the dynamics of two qubits map in a three complex variable representation.

In the following, we shall work in the product basis of qubits, i.e. set $|e_k\rangle = |i\rangle|j\rangle$ $(i,j \in \{0,1\})$. One step of the purification protocol $\mathscr{F}$ is a holomorphic endomorphism which depends on three complex variables

$$\mathscr{F}: \quad z_1 \mapsto \frac{a^2b^2+z_1^2+ab(z_2^2+z_3^2)}{1+a^2z_1^2-a(z_2^2+z_3^2)}, \quad z_2 \mapsto \frac{a(b-z_1^2)+z_2^2-a^2bz_3^2}{1+a^2z_1^2-a(z_2^2+z_3^2)}, \quad z_3 \mapsto \frac{a(b-z_1^2)-a^2bz_2^2+z_3^2}{1+a^2z_1^2-a(z_2^2+z_3^2)}, \tag{13}$$

where $a = \tan x e^{\imath\psi}$, $b = e^{-2\imath\psi}$.

Let us study the case $a = 0$ i.e. $x = 0$ (no unitaries). Then the map

$$\mathscr{F}: z_1 \mapsto z_1^2,\ z_2 \mapsto z_2^2,\ z_3 \mapsto z_3^2 \tag{14}$$

has $(2n)^3$ fixed points of the n-th order: $\mathbf{z} = (z_1, z_2, z_3)$, $z_i \in \{0, e^{2\imath\pi k_i/(2n-1)}\}$, from those the $(0,0,0)$ is the stable one and all the remaining ones are unstable. Although, they are outside the domain of $\mathscr{F}$, we can regard the points with one component equal to ∞ and the rest being 0 as additional fixed points of the map. Physically, these points correspond to the first three states of product basis, however, their stability can only be studied in a different choice of basis $\{|e_i\rangle\}$. The Julia set of the map (14) contains the surface of $S(\mathbf{0},1)$ in $\hat{\mathbb{C}}^3$. The points inside $S(\mathbf{0},1)$ converge to $(0,0,0)$ i.e. system converges to $|1\rangle|1\rangle$. The points outside $S(\mathbf{0},1)$ go to ∞.

The case when $a \to \infty$ i.e. $x \to \pi/2$ (off-diagonal unitary) is described by the map

$$\mathscr{F}: z_1 \mapsto \frac{b^2}{z_1^2},\ z_2 \mapsto -\frac{bz_3^2}{z_1^2},\ z_3 \mapsto -\frac{bz_2^2}{z_1^2}. \tag{15}$$

Fixed points of the first order of (15) are $(\pm b, 0, 0)$, $(\pm b, -b, -b)$, $(\pm b, (-1)^{\frac{1}{3}}b, -(-1)^{\frac{2}{3}}b)$, $(\pm b, -(-1)^{\frac{2}{3}}b, (-1)^{\frac{1}{3}}b)$, from those only $(\pm b, 0, 0)$ are stable, when $|b| > 2$.

The map for the case when $a = 1,\ b = 1$ i.e. $x = \pi/4,\ \phi = 0$ (symmetric unitary) can be expressed as

$$\mathscr{F}: \quad z_1 \mapsto -1-\frac{2\left(1+z_1^2\right)}{-1-z_1^2+z_2^2+z_3^2}, \quad z_2 \mapsto 1+\frac{2\left(z_1^2-z_2^2\right)}{-1-z_1^2+z_2^2+z_3^2}, \quad z_3 \mapsto -1+\frac{2\left(-1+z_2^2\right)}{-1-z_1^2+z_2^2+z_3^2}. \tag{16}$$

Numerical calculations prove the existence of fifteen fixed points of the first order, out of those only $(1,0,0)$ is stable. This also gives a stable fixed point of the purification protocol, $|\Psi_3\rangle$.

The analytic approach for one complex variable is well known for holomorphic endomorphisms, especially for rational maps, see [6] and [7]. For two complex variables, a general analysis exists for holomorphic automorphisms, especially for Hénon map, see [6]. For three complex variables a detailed analysis of the dynamics is missing. For the two complex variable map there exist a lemma (see [6]) which describes precisely the behavior of any invertible, holomorphic map, depending only on the eigenvalues of the map. For special cases a similar lemma can be formulated in the case of three complex variables.

Lemma. *Let $\mathscr{F}$ be an invertible, holomorphic map with stable fixed point* $\mathbf{a} \in \mathbb{C}^3$. *Suppose that the eigenvalues λ, μ, ν of $J(\mathscr{F})(\mathbf{a})$ satisfy the condition* $0 < |\mu| \leq |\nu| \leq |\lambda| < 1$. *Then by the change of local coordinates $\mathscr{F}$ in iteration limit converges uniformly in the neighborhood of* $\mathbf{0}$ *to the map $\mathscr{I}$ where*

$$\mathscr{I} : (x,y,z) \mapsto (\lambda x, \mu y + g(x,z), \nu z + h(x,y)),$$

g, h are holomorphic. Let us define

$$\mathscr{I}_{\lambda,\mu,\nu} : (x,y,z) \mapsto (\lambda x, \mu y, \nu z)$$
$$\mathscr{I}_{\lambda,k,\nu} : (x,y,z) \mapsto (\lambda x, \lambda^k(y + x^k), \nu z).$$

If $g(x,z) \equiv g(x)$ and $h(x,y) \equiv h(x)$ then

$$\text{if } \forall k,l : \lambda^k \neq \mu \wedge \lambda^l \neq \nu \text{ then } \mathscr{I} = \mathscr{I}_{\lambda,\mu,\nu},$$
$$\text{if } \exists k : \lambda^k = \mu \wedge \forall l : \lambda^l \neq \nu \text{ then } \mathscr{I} = \mathscr{I}_{\lambda,k,\nu}.$$

Proof. Firstly, it can be seen that the behavior of $\mathscr{F}$ in iteration limit converges on a sufficiently small neighborhood U of $\mathbf{a}$ to the mapping $(x,y,z) \mapsto (\lambda x,\ g(x,y,z),\ h(x,y,z))$, where g, h are holomorphic. Moreover, $\mathscr{F}$ can be reduced to $(x,y,z) \mapsto (\lambda x,\ \mu y + \bar{g}(x,z),\ \nu z + \bar{h}(x,y))$, where $\bar{g}$, $\bar{h}$ are holomorphic.

Let $\mathscr{F} \equiv (f,\ g,\ h)$. Let us deal with the sequence $\{\frac{1}{\lambda^n} f^{\circ n}(\mathbf{p})\}$, $\mathbf{p} = (p_x, p_y, p_z) \in U$. One can prove that $\{\frac{1}{\lambda^n} f^{\circ n}(\mathbf{p})\} \rightrightarrows \psi(\mathbf{p})\}$ on U and $\psi(\mathbf{p}) = p_x$. Then both sides of $\{\frac{1}{\lambda^n} f^{\circ n}(\mathscr{F}(\mathbf{p}))\} = \lambda\{\frac{1}{\lambda^{n+1}} f^{\circ n+1}(\mathbf{p})\}$ converge uniformly to $\psi(\mathscr{F}(\mathbf{p})) = \lambda\psi(\mathbf{p})$. This proves the first step i.e. $\mathscr{F} : (x,y,z) \mapsto (\lambda x,\ g(x,y,z),\ h(x,y,z))$.

Secondly, let us translate the origin of the coordinates to $\mathbf{a}$, then $\mathscr{F}$ can be expanded into Taylor series in each coordinate and in iteration limit $\mathscr{F}$ behaves as $(x,y,z) \mapsto (\lambda x, \mu y + \tilde{g}(x,y,z), \nu z + \tilde{h}(x,y,z)$. One can prove that $\{\frac{1}{\mu^n}\frac{\partial g^{\circ n}}{\partial y}\}$ and $\{\frac{1}{\nu^n}\frac{\partial h^{\circ n}}{\partial z}\}$ converge uniformly on U towards 1, consequently $\mathscr{F}$: $(x,y,z) \mapsto (\lambda x, \mu y + \bar{g}(x,z), \nu z + \bar{h}(x,y))$.

To prove the final tuning of $\mathscr{F}$, one need to find out the coordinates where $\mathscr{F}$ behaves as $\mathscr{I}_{\lambda,\mu,\nu}$ or $\mathscr{I}_{\lambda,k,\nu}$. For the two complex variables case it was proved in [6] that the final tuning depends on the value of the power of λ with respect to the remaining eigenvalues. For our $\mathscr{F}$ the situation is more complicated: $\bar{g}$ and $\bar{h}$ mix the values of coordinates. One can study less complicated cases in the sense of [6] and tune $\mathscr{F}$ when $\bar{g}(x,z) \equiv \bar{g}(x)$ and $\bar{h}(x,z) \equiv \bar{h}(x)$. One can find out local coordinates η_1, η_2 in the forms $\eta_1(p) \equiv (x, q_1(x), z)$ and $\eta_2(p) \equiv (x, y, q_2(x))$ such that $\mathscr{F}$ has desired form in the new coordinates $\eta_1 \circ \eta_2$. □

According to the lemma a general invertible, holomorphic map $\mathscr{F}$ in the iteration limit behaves as linear in the direction corresponding to the basis vector associated with the largest eigenvalue. In the remaining directions, in general, the behavior is different. For special cases when the map depends in these directions only on two specific variables, we have linear or exponential behavior in these directions as well.

CONCLUSION

We focused on a special type of nonlinear maps used for quantum information processing - the purification protocols. Complex chaos is known to be present for such protocols in the one-qubit case (proven analytically) and found numerically for the case of more qubits. Extreme sensitivity to initial settings can destroy the functionality of the protocol which has serious practical implications. We presented an analytic approach to the two qubits case using a special parametrization of the input states. We found the corresponding explicit form of the map which is the first step in the analytic study of the properties of the dynamics of purification protocols involving multi-partite systems.

ACKNOWLEDGMENTS

This work was supported by the Czech and Hungarian Ministries of Education (MEB 040805, CZ - 10/2007), by MSM 6840770039 and GAČR 202/08/H072 and by the Hungarian Scientific Research Fund (T049234 and T068736).

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