

## Measurement-Induced Chaos with Entangled States

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The dynamics of an ensemble of identically prepared two-qubit systems is investigated which is subjected to the iteratively applied measurements and conditional selection of a typical entanglement purification protocol. The resulting dynamics exhibits strong sensitivity to initial conditions. For one class of initial states two types of islands characterize the asymptotic limit. They correspond to a separable and a fully entangled two-qubit state, respectively, and their boundaries form fractal-like structures. In the presence of incoherent noise an additional stable asymptotic cycle appears.

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*Introduction.*—Entanglement is at the heart of quantum physics since its discovery [1]. However, it was only recently that the focus has been put on entanglement as a resource for quantum communication and quantum information processing [2]. Various protocols have been developed to detect [3], generate, and distill entanglement [4]. From an ensemble of identical quantum states, one can produce an ensemble yielding a higher degree of entanglement by unitary transformations, measurements, and selection conditioned on measurement outcomes. These nonlinear processes are referred to as entanglement distillation or purification. Entanglement distillation protocols play a crucial role in increasing the quality of communication channels and have also been used to define the degree of entanglement in an operational sense.

The phenomenon that sensitivity to initial conditions leads to chaotic dynamics in classical physics is well-known. Similar phenomena in closed quantum systems are, however, excluded by the quantum unitary evolution [5]. In the case of open quantum systems the restriction to unitarity is lifted. The evolution of an open quantum system is sometimes pictured as additionally having an environment that performs generalized measurements on it. In general, any type of measurement makes the evolution nonunitary. Entanglement purification, i.e., selection of certain systems from an ensemble of identical systems based on the results of partial measurements, can be regarded as a generalized feedback mechanism. The nonlinear dynamics resulting from such generalized feedback has been shown to lead in certain cases to a chaos [6]. This type of chaos is essentially different from that arising from the stochastic dynamics of a continuously measured open quantum system [7], which can become chaotic in the semiclassical regime while still showing signatures of quantum behavior [8].

The generalized feedback resulting from measurement-based selection plays a crucial role in the case of various entanglement purification protocols [4]. Applying the purification protocol [9] on an ensemble of single qubits prepared in identical pure states, after each iteration step the remaining, selected ensemble of qubits will again be in identical, pure quantum states. The dynamical evolution can be characterized by a rational nonlinear map over the extended complex plane representing the pure states. This map has been proven to lead to truly chaotic behavior [6]. A single qubit is the simplest quantum system, and also lacks all genuine nonlocal quantum properties. Thus the above mentioned study left a fundamental question open, which in turn rises naturally in the context of entanglement purification: Can we find sensitivity to initial conditions for genuine multipartite quantum properties, in particular, for entanglement, if only local operations and classical communication are applied?

In this Letter we focus on a particular entanglement purification protocol [9], and demonstrate the existence of true chaos which manifests also in the evolution of entanglement. An important feature of the protocol is that it maps pure states onto pure states; moreover, it may also increase purity of initially mixed states. Our analysis is accomplished by showing that the convergence to a fully entangled or a separable asymptotic attractor can be sensitive to the initial state.

*Measurement-based nonlinear dynamics.*—The pure quantum state of a system consisting of a pair of qubits can be expressed in the computational basis as  $|\Psi\rangle = c_1|00\rangle + c_2|01\rangle + c_3|10\rangle + c_4|11\rangle$ . We consider an ensemble of qubit pairs prepared in the state  $|\Psi\rangle$  as the input to the entanglement purification protocol [9] schematically depicted in Fig. 1. This protocol realizes a nonlinear transformation of the quantum state  $|\Psi\rangle$  according to  $c_i \xrightarrow{S} N c_i^2$ , where  $N$  is a necessary normalization factor.

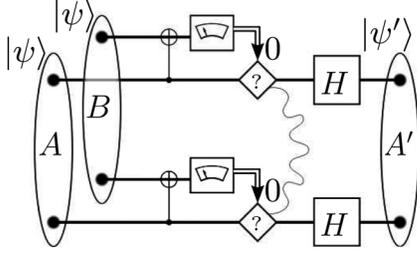


FIG. 1. Schematic representation of one iteration step of the entanglement purification process. The diamond shaped elements denote acceptance of the pair of qubits only when both measurements yield 0. The depicted procedure is used to prepare from an ensemble  $\mathcal{E}$  made up of pairs of qubits all in a state  $|\psi\rangle$ , an ensemble  $\mathcal{E}'$  made up of pairs of qubits in the state  $|\psi'\rangle$ . During the procedure, half of the pairs in  $\mathcal{E}$  are completely used up, while a portion of the other half is retained depending on the measurement outcome. Here we depicted the special case  $U = H \otimes H$ , where  $H$  is the Hadamard gate.

The nonlinearity is due to the generalized feedback realized by the measurement-based conditional selection. By following each nonlinear transformation by a unitary  $U$ , further nontrivial dynamics can be generated. A complete iteration can be expressed as the transformation  $|\Psi'\rangle = US|\Psi\rangle$ . Note, that only local unitary transformations comply with the concept of entanglement purification. For the following analysis we fix the unitary transformation to be  $U = H \otimes H$ , where  $H$  is a one-qubit Hadamard gate [ $H_{ij} = (-1)^{ij}/\sqrt{2}$ ].

For Hilbert spaces of dimensionalities of more than two, such as for that of a two-qubit system, the representation of a pure state requires a vector of several complex numbers, and the nonlinear dynamics is described by a nonlinear map on this higher dimensional complex space. The mathematics of nonlinear dynamical maps in several complex variables is substantially more involved and much less understood than the same for a single complex variable. Even the existence of chaotic regions is a nontrivial question [10].

We are interested in the evolution of genuine multipartite properties of the system under the iterative nonlinear dynamics. Since this protocol describes an entanglement purification protocol, of all such properties, entanglement is of the most concern. Thus we consider states parameterized by the complex number  $\zeta$  ( $\zeta \in \hat{\mathbb{C}}$  including infinity) of the form

$$|\Psi(\zeta)\rangle = \mathcal{N}(\zeta)(|00\rangle + \zeta|11\rangle), \quad (1)$$

where the normalization factor reads  $\mathcal{N}(\zeta) = (1 + |\zeta|^2)^{-1/2}$ . The degree of entanglement of this state is completely determined by  $\zeta$  via the binary entropy function  $H(\mathcal{N}(\zeta)^2)$  [11]. By studying the evolution of these states we can learn about the general multipartite properties of this protocol. The analytic treatment of this evolution is greatly simplified by the fact that the states of

the form of Eq. (1) are invariant under two successive iterations, in particular, yielding

$$|\Psi(\zeta)^{(2)}\rangle = \mathcal{N}(g(\zeta))(|00\rangle + g(\zeta)|11\rangle), \quad (2)$$

where  $g(\zeta) = (2\zeta^2)/(1 + \zeta^4)$ . Thus the description of the dynamics for this class of initial states simplifies to a nonlinear map of a single complex variable. The function  $g(\zeta) : \hat{\mathbb{C}} \mapsto \hat{\mathbb{C}}$  is a fourth order rational function generating a one variable complex dynamical map on the Riemann sphere. The map is chaotic in the sense that the corresponding Julia set is nonvacuous [12]. Fourth order maps in general can lead to rather involved behavior. We base our analysis on the observation that the fourth order rational map can be written as a composition of a second order rational function with itself  $g(\zeta) = f(f(\zeta)) = f^{\circ 2}(\zeta)$ , where

$$f(\zeta) = \frac{1 - \zeta^2}{1 + \zeta^2}. \quad (3)$$

Moreover, the same function also describes the quantum state after an odd number of iterations, according to  $|\Psi(\zeta)^{(2n+1)}\rangle = \mathcal{N}[|00\rangle + |11\rangle + f^{\circ 2n+1}(\zeta)(|01\rangle + |10\rangle)]$ . Thus the iterative dynamics of initial states of the form of Eq. (1) is equivalent to the iterative dynamics generated by the second order rational map defined in Eq. (3). In the following we examine this mapping in some detail.

*Stable cycles of the dynamics.*—The long-term behavior of a rational map can be analyzed [12] by following the orbits of its critical points [points where the derivative of the map vanishes  $f'(\zeta_c) = 0$ ]. The second order map in Eq. (3) has two critical points:  $\zeta_{c1} = 0$  and  $\zeta_{c2} = \infty$ . The first one,  $\zeta_{c1}$ , is part of the superattracting cycle  $\{0, 1\}$ , while the second one lands on the same cycle after two iterations. Therefore, the map has one stable (superattractive) cycle. Translated back to the language of states, we have to take into account that the function  $f$  describes a state in the same form only in every second step. Thus, depending on the parity of steps when reaching the first element of the cycle, we have two distinct cases.

In the first case, when  $f^{\circ 2n}(\zeta) = 1$ , the corresponding state reads

$$|\Psi^{(2n)}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = |\Phi^+\rangle. \quad (4)$$

After an even number of steps, one reaches a fully entangled state, the Bell state  $|\Phi^+\rangle$ . Since this state is invariant under both the nonlinear transformation  $S$  and the unitary operation  $H \otimes H$ , we have  $|\Psi^{(2n)}\rangle = |\Psi^{(2n+1)}\rangle = |\Phi^+\rangle$ . Therefore, the Bell state  $|\Phi^+\rangle$  is an asymptotically stable fixed point of the dynamics.

In the second case, when  $f^{\circ 2n}(\zeta) = 0$ , the corresponding state reads

$$|\Psi^{(2n)}\rangle = |00\rangle. \quad (5)$$

The above product state is not invariant under the iterative dynamics; however, any subsequent step will leave the state completely separable. In particular, after an odd number of steps we find

$$|\Psi^{(2n+1)}\rangle = \frac{1}{2}(|0\rangle + |1\rangle) \otimes (|0\rangle + |1\rangle). \quad (6)$$

Therefore, the second stable cycle is of length two; both of its members are separable pure states. Because of a theorem on rational maps [12], the degree of the rational function determines the maximum number of stable cycles. For a rational map of degree two, at most two stable cycles can exist. In our case, the single stable cycle of the rational map  $f$  can lead to two different stable cycles of the dynamics, depending on the parity of the number of steps when approaching the first element of the cycle. Thus, we have found all possible stable cycles of the dynamics restricted to initial pure states of the form (1).

*Sensitivity to initial states.*—The two stable cycles of the dynamics are very different. One of them is a single, completely entangled pure state, a Bell state, while the other one is an oscillation between two separable pure states of the two qubits. We ask now the question, what are the initial states converging to each of the stable cycles?

Let us first discuss the case of real values for the parameter  $\zeta$ . The function  $f$  maps real numbers to real numbers; thus, it can be restricted to  $\mathbb{R}$ . The members of the stable cycle  $\{0, 1\}$  are also real numbers. We can now determine the basin of attraction for the two cases of convergence, i.e., convergence to 0 after an even or an odd number of steps, which we shall call even-zero or odd-zero convergence, respectively. The immediate neighborhood of the fixed point 0, belonging to even-zero convergence, is determined by the equation  $|f^{\circ 2}(\zeta)| < |\zeta|$ , with the condition  $|\zeta| < 1$ . The corresponding equation can be explicitly solved yielding  $|\zeta| < \zeta_A$  where  $\zeta_A = (a - 1 - 2/a)/3$  with  $a = (17 + 3\sqrt{33})^{1/3}$ . The preimages of the interval  $(-\zeta_A, \zeta_A)$  belong to an odd-zero convergence region. By solving the corresponding equation we find two distinct intervals of odd-zero convergence  $\zeta_A < |\zeta| < \zeta_B$  where  $\zeta_B = \sqrt{(-2 + 2a + a^2)/(2 + 4a - a^2)}$ . It is easy to see that the region  $|\zeta| > \zeta_B$  is mapped after one iteration to the region  $\zeta_A < |\zeta| < \zeta_B$ ; thus, it belongs to even-zero convergence. To summarize the behavior of the map restricted to the reals, we have found that regions of odd-zero and even-zero convergence follow each other; these open sets belong to the Fatou set. The border point  $\zeta_A$  is a repulsive fixed point of the map, while the other border points  $\{-\zeta_A, -\zeta_B, \zeta_B\}$  are preimages of  $\zeta_A$ . These four points belong to the Julia set.

A similar analysis can be repeated for the map  $f$  with domain  $\hat{\mathcal{C}}$ . The preimages of 0 with their open small neighborhoods provide regions of odd-zero or even-zero convergence, forming the Fatou set. The Fatou set is an open set, while the complementary closed set is the Julia set. A small neighborhood of the origin will belong to even-zero convergence; a sufficient condition for this is

that  $|f(f(\zeta))| < |\zeta|$ . From this condition we can determine the maximum radius  $\zeta_C$  of a circle around the origin belonging to even-zero convergence. The radius  $\zeta_C$  can be calculated explicitly from an algebraic equation; its numerical value is  $\zeta_C \approx 0.475$ . The first order preimages of 0 are 1 and  $-1$ . Thus, 1 and  $-1$  together with a region around them belong to odd-zero convergence. The preimage of the circle of convergence around zero determines an immediate region of convergence around 1 and  $-1$ . In a similar manner one can continue this process and determine the next order preimages and regions of convergence around them. The numerically calculated convergence regions are shown in Fig. 2. We can clearly recognize a chessboardlike structure of light (green) and dark (blue) colored islands, belonging to convergence to fully entangled or completely separable states, respectively. The islands follow a self-similar structure with decreasing size. A Julia set is formed by the border points between the two colors, with no stable cycles in it. The asymptotic entanglement of the two-qubit system behaves chaotically; it is sensitive to the initial state on arbitrary small scales. Since this is an asymptotic map, it involves an infinite-in-time limit process.

*Mixed initial states*—Up to now, we have considered pure initial states. Adding noise to the initial state, described by a density operator, will alter the dynamics. We will test the sensitivity to a certain type of noise by adding the unit matrix to the density matrix representing the initial pure state

$$\rho(\zeta, \lambda) = \lambda |\Psi(\zeta)\rangle\langle\Psi(\zeta)| + \frac{1-\lambda}{4} \mathbb{1}, \quad (7)$$

where  $\lambda \in [0, 1]$ , and  $\mathbb{1}$  stands for the unit operator acting on the Hilbert space of the two qubits. Since the dynamics

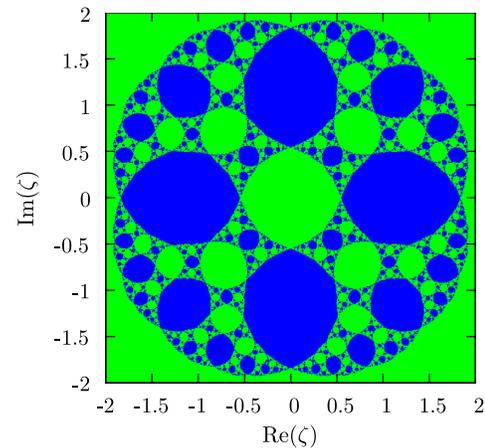


FIG. 2 (color online). Convergence of initial states of Eq. (1) to the two limiting cycles as a function of the complex parameter  $\zeta$ . The dark (blue) color denotes convergence to the maximally entangled state of Eq. (4), and the light (green) color the convergence to the cycle of the two separable states in Eqs. (5) and (6).

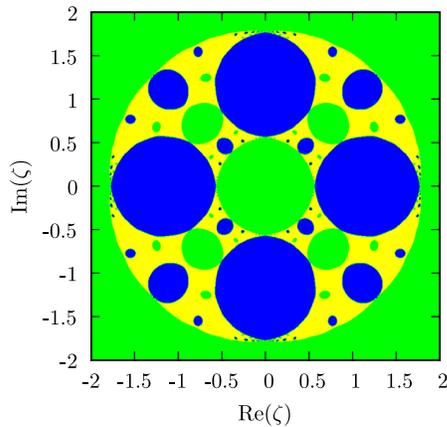


FIG. 3 (color online). Convergence of mixed initial states  $\rho(\lambda, \zeta)$  of Eq. (7) to the three limiting cycles as a function of the complex parameter  $\zeta$ , at  $\lambda = 0.75$ . The darkest (blue) and second-darkest (green) colors denote convergence to the same cycles as in Fig. 2, while the lightest (yellow) color denotes convergence to the separable mixed cycle in Eq. (8). As  $\lambda$  is decreased from 1 to 0, the corresponding convergence plots gradually turn from those in Fig. 2 to uniform lightest (yellow) color.

is no longer restricted to pure states, we can expect that further stable fixed cycles will appear, containing mixed states. The stability of the fixed cycles can be proven in any convenient representation. We chose the Fano representation where the real expansion coefficients with respect to the 16 generalized Pauli matrices represent an arbitrary density matrix [13]. Moreover, the Fano representation is convenient for numerical simulation of the dynamics also. By calculating the eigenvalues of the Jacobian in the Fano representation for each numerically found cycle, we concluded that among them the only stable cycle is the length two cycle  $\{\rho_1, \rho_2\}$ , where

$$\begin{aligned} \rho_1 &= \frac{1}{2}(|00\rangle\langle 00| + |11\rangle\langle 11|), \\ \rho_2 &= \frac{1}{4}[|00\rangle\langle 00| + |11\rangle\langle 11| + (|01\rangle + |10\rangle)(\langle 01| + \langle 10|)]. \end{aligned} \quad (8)$$

The same calculation for the cycles known from the pure initial state case indicated their stability against perturbation by arbitrary mixed states. In Fig. 3 we show the convergence towards the stable cycles of the mixed state dynamics. The third, mixed stable cycle  $\{\rho_1, \rho_2\}$  denoted by the lightest (yellow) color washes out the fine structure of the pure state picture. Islands of purification towards both fully entangled and completely separable states remain visible even for the value  $\lambda = 0.75$  of the mixing parameter. Understanding the structure of convergent areas requires further studies. An interesting question is whether

the number of the islands purifying to entangled states is finite or it possesses a fractal structure.

*Conclusions.*—We have demonstrated that entanglement in a quantum system can evolve truly chaotically, exhibiting sensitivity to the initial condition. Our results give an insight into the properties of the general pure state dynamics of a protocol that is described by a nonlinear dynamical map in three complex variables. We have found that extending the space of initial states to a special class of mixed states, a new, mixed attracting cycle appears. It would require further studies to decide whether adding other types of noise to the initial state would introduce additional attractors.

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