

## QUANTUM TELEPORTATION WITH PAIR-COHERENT STATES

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Recently, it has been argued that all presently performed continuous variable quantum teleportation experiments could be explained using a local hidden variable theory. In this paper, we study a modification of the original protocol which requires a fully quantum mechanical explanation even when coherent states are teleported. Our calculations of the fidelity of teleportation using a pair-coherent state under ideal conditions suggest that fidelity above the required limit of  $1/2$  may be achievable in an experiment also.

*Keywords:* Quantum teleportation; local realism; pair-coherent state.

### 1. Introduction

In their recent paper Caves *et al.*<sup>1</sup> argued that the process of continuous variable teleportation<sup>2,3</sup> of Gaussian states allows for a local hidden variable description. Earlier works<sup>4,5</sup> focused on the necessity of entanglement for teleportation, and found that fidelities up to  $\mathcal{F}_{av} = 1/2$  can be achieved even with no entanglement.

In the present paper we focus on a teleportation scheme that does not allow for a local hidden variable description. In Ref. 1, it has been suggested to use non-Gaussian states at the input. For this case, an upper limit  $\mathcal{F}_{av} = 2/3$  on the fidelity has been established, below which there may exist a local hidden variable description.

Experimental realization of such a scheme involving bright non-Gaussian input states, however, does not seem formidable in the near future. The underlying problem is that to achieve such high fidelity, we would require high entanglement, and therefore high photon numbers for both the entangled resource and the input.

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On the other hand, replacing the entangled resource with a non-Gaussian state can also render the local realistic description impossible. In the present paper we propose to use a pair-coherent<sup>6,7</sup> state as an entangled resource. In contrast to the two-mode squeezed vacuum for which measurement of commuting quadrature operators do not violate local realism,<sup>8-10</sup> such measurements on a pair-coherent state cannot be described by a local hidden variable theory.<sup>11</sup> And since the teleportation protocol calls for measurement of commuting quadratures, this replacement is necessary to rule out a local realistic description of teleportation of Gaussian states.

The use of pair-coherent states seems viable due to recent developments in experiments with non-degenerate optical parametric oscillators<sup>12,13</sup> (NOPO), which have been shown theoretically to produce pair-coherent states under certain conditions.<sup>14,15</sup> Since the pair-coherent state is non-Gaussian, the negative part of its Wigner function has non-zero support, and this is enough to eliminate the possibility of a local hidden variable description of the teleportation process.

The structure of our paper is as follows: In Sec. 2, we briefly recall the definition and important properties of pair-coherent states. In Sec. 3, we overview our formalism used to describe the teleportation process and present the results on the teleportation fidelities of coherent states. In Sec. 4, we interpret our results in the light of the earlier requirements, and we finally conclude in Sec. 5.

## 2. Pair-Coherent States

Pair-coherent states<sup>6</sup> of the two mode electromagnetic field are simultaneous eigenstates of the pair-annihilation operator  $ab$  and the difference of the number operators  $a^\dagger a - b^\dagger b$ , i.e.

$$ab|\zeta, q\rangle = \zeta|\zeta, q\rangle, \quad \text{and} \quad a^\dagger a - b^\dagger b|\zeta, q\rangle = q|\zeta, q\rangle. \quad (1)$$

These states are important in quantum information processing, because they are the only non-Gaussian entangled states<sup>16</sup> that could be prepared reliably sometime in the near future.<sup>12-15</sup>

Pair-coherent states may either be written using the Fock basis as

$$|\zeta, q\rangle = \frac{1}{\sqrt{\zeta^q I_0(2\zeta)}} \sum \frac{\zeta^n}{\sqrt{n!(n+q)!}} |n+q, n\rangle, \quad (2)$$

or in a coherent state representation as

$$|\zeta, q\rangle = \frac{e^\zeta}{2\pi\sqrt{\zeta^q I_0(2\zeta)}} \int_0^{2\pi} \left(\sqrt{\zeta}e^{i\vartheta}\right)^{-q} |\sqrt{\zeta}e^{i\vartheta}\rangle |\sqrt{\zeta}e^{-i\vartheta}\rangle d\vartheta, \quad (3)$$

where  $I_0$  denotes the modified Bessel function of the first kind. Since Eq. (3) is essentially a contour integral on a zero centered circle of the complex plane, pair-coherent states are also sometimes referred to as “circle” states.

In the present paper we make use of two features of pair-coherent states. First is that they are entangled whenever  $\zeta > 0$ , as can easily be seen from Eq. (2), which

happens to be a valid Schmidt decomposition. The other is that these states are also non-Gaussian with their Wigner functions exhibiting significant negativity:

$$W(\alpha/2; \beta/2) = \frac{4 \exp(-|\alpha|^2 - |\beta|^2)}{\pi^2 I_0(2\zeta)} \sum_{m,n=0}^{\infty} \cos[(m-n)(\varphi_\alpha + \varphi_\beta)] \frac{(\zeta|\alpha||\beta|)^{m+n}}{(m!)^2(n!)^2} \\ \times {}_2F_0(-m, -n; ; -1/|\alpha|^2) {}_2F_0(-m, -n; ; -1/|\beta|^2), \quad (4)$$

where  $\varphi_\alpha$  and  $\varphi_\beta$  denote the argument of  $\alpha$  and  $\beta$ , respectively. (The generalized hypergeometric function  ${}_2F_0$  represents only a finite sum.)

### 3. Teleportation of Coherent States

Now we turn to the modified quantum teleportation protocol, where the two-mode squeezed vacuum constituting the quantum channel has been replaced by a pair-coherent state  $|\zeta, 0\rangle$ . Comparing Eq. (4) with Eqs. (2) and (3), we find that the description of pair-coherent states is more convenient in the Hilbert space formalism. Indeed, when the input states are coherent states, the coherent state representation (3) turns out to be the most efficient.

We use the coherent state description of Bell state measurement from Ref. 17. We write the state corresponding to Alice's homodyne measurement outcome  $A = (X + iP)/\sqrt{2}$  as

$$|B(X, P)\rangle = \int_{\mathbb{C}} e^{A\gamma^* - A^*\gamma} |\gamma + A\rangle |\gamma^* - A^*\rangle d^2\gamma. \quad (5)$$

The classical information sent to Bob corresponds to this complex value  $A$ .

We construct the transfer operator following Ref. 18 using this expression. To obtain the transfer operator in general terms, we first note that the Bell states possess the property

$$|B(X, P)\rangle = D(A) \otimes D(-A^*) |B(0, 0)\rangle = D(2A) \otimes \mathbb{1} |B(0, 0)\rangle, \quad (6)$$

which together with  ${}_1\langle\psi|B(0, 0)\rangle_{12} = \mathcal{I}|\psi\rangle_2$  imply

$${}_2\langle\psi|B(X, P)\rangle_{12} = \mathcal{I}D(2A)|\psi\rangle_1, \quad (7)$$

where  $\mathcal{I}$  is the anti-linear operator primitive defined as  $\mathcal{I}|n\rangle = |n\rangle$ . Since this partial scalar product describes Alice's measurement, the anti-linearity enters into the expression of the transfer operator:

$$T_\zeta(A) := \frac{e^\zeta}{2\pi\sqrt{\pi}I_0(2\zeta)} \int_0^{2\pi} |\sqrt{\zeta}e^{i\vartheta}\rangle \langle\sqrt{\zeta}e^{i\vartheta}| D(-2A) d\vartheta. \quad (8)$$

The teleportation procedure can be written as follows. Our initial state is  $|\psi_0\rangle_{123} = |\psi_{\text{in}}\rangle_1 |\zeta, 0\rangle_{23}$ , and the transfer operator gives the unnormalized outcome

$$|\psi_t(A)\rangle = T_\zeta(A)|\psi_{\text{in}}\rangle \quad (9)$$

of the first stage of the teleportation — before Bob's displacement. We note that this definition is slightly different from that of Ref. 18. The probability

distribution for Alice's measurement is given by the norm  $P(A) = \|\psi_t(A)\|$ . Hence the actual output state after the displacement can be written  $|\psi_{\text{out}}(A)\rangle = 1/\sqrt{P(A)}D(\beta)T_\zeta(A)|\psi_{\text{in}}(A)\rangle$  if we assume that Bob attempts to reconstruct the state with a coherent displacement  $\beta$ . We use the definition of average fidelity

$$\mathcal{F}_{\text{av}} = \int_{\mathbb{C}} \mathcal{F}(A)P(A) d^2A, \quad (10)$$

where  $\mathcal{F}(A) = |\langle\psi_{\text{in}}|D(\beta)T_\zeta(A)|\psi_{\text{in}}\rangle|^2/P(A)$  is the fidelity of a single teleportation event. In the following, we shall take  $\beta = g2A$ , and assume that the gain factor  $g$  may be adjusted by Bob to improve the fidelity of teleportation.

Now let the input state be  $|\psi_{\text{in}}\rangle = |\alpha\rangle$ , a coherent state. According to Eq. (9), after Alice's measurement our pseudo output state is

$$|\psi_t(A)\rangle = \frac{e^\zeta}{2\pi\sqrt{\pi}I_0(2\zeta)} \int_0^\infty \langle\sqrt{\zeta}e^{i\vartheta}|D(-2A)|\alpha\rangle|\sqrt{\zeta}e^{i\vartheta}\rangle d\vartheta, \quad (11)$$

which gives for the average fidelity the expression

$$\begin{aligned} \mathcal{F}_{\text{av}}(\alpha, \zeta, g) &= \frac{e^{-\frac{(g-1)^2}{1+g^2}|\alpha|^2}}{4(1+g^2)I_0(2\zeta)} \sum_{m,n=0}^\infty \frac{\zeta^{m+n}}{(m!n!)^2} \sum_{p=0}^m \sum_{q=0}^n \binom{m}{p} \binom{n}{q} (-1)^{m+n+p+q} \\ &\quad \times \sum_{j=0}^{\min\{n+p, m+q\}} \binom{n+p}{j} \binom{m+q}{j} j! \left(\frac{(1-g)^2}{1+g^2}\right)^{m+n-j} \\ &\quad \times \frac{g^{m+n+2(p+q-j)}}{(1+g^2)^{p+q}} (|\alpha|^2)^{m+n-j}. \end{aligned} \quad (12)$$

In the special case  $g = 1$ , this simplifies to

$$\mathcal{F}_{\text{av}}(\alpha, \zeta, 1) = \frac{1}{2I_0(2\zeta)} \sum_{n,m=0}^\infty \frac{(\zeta/2)^{m+n}(m+n)!}{(m!n!)^2}, \quad (13)$$

and hence becomes independent of the input coherent amplitude  $\alpha$ . This fidelity is compared with that of the original protocol on Fig. 2. On Fig. 1, we can observe that the effect of gain tuning is similar to that of on the original protocol: the optimal gain is always  $g_{\text{opt}} < 1$ , and it approaches 1 in the limit of large amplitudes.

#### 4. Discussion

An interesting feature of the pair-coherent state quantum channel is that the fidelity does not always increase with the entanglement, which is reflected in a maximum of the fidelity as a function of  $\zeta$ . The likely reason for this is that the Bell measurement consisting of the beam-splitter and the homodyne detectors is not the matching<sup>19</sup> measurement for the pair-coherent state. This is in great contrast with the two-mode squeezed vacuum, for which this measurement becomes matching in the limit of infinite squeezing. As indicated by the numerical results, the parameters are

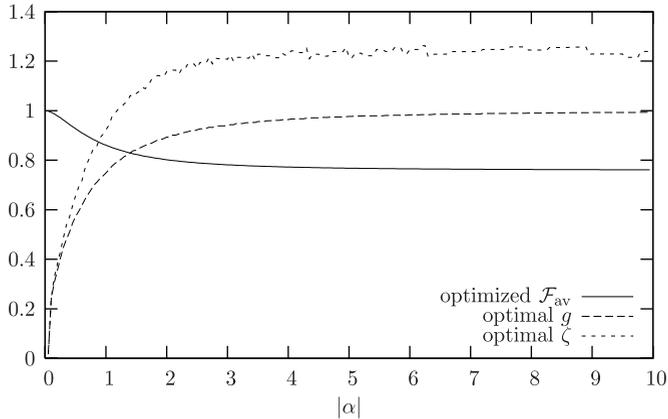


Fig. 1. Optimized average fidelity plotted against the input state  $\alpha$ . Also plotted are the optimal values of  $\zeta$  and  $g$ . The plots demonstrate that the optimal gain  $g \rightarrow 1$  as  $|\alpha| \rightarrow \infty$ , hence  $\zeta \rightarrow 1.2357$  approximately in the same limit.

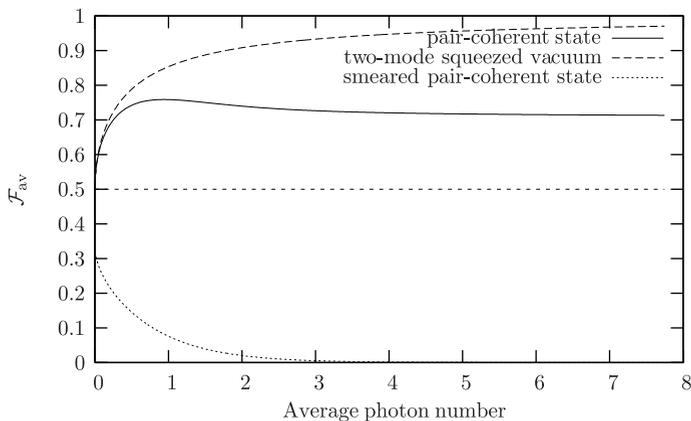


Fig. 2. Average fidelities for the teleportation of coherent states using a two-mode squeezed state and a pair coherent state, when the gain tuning  $g = 1$ . In both cases the average fidelity is independent of the input coherent state  $|\alpha\rangle$ . Also plotted is the fidelity of teleportation when using a “smeared-out” pair-coherent state.

probably the closest to the matching conditions when the pair-coherent state has  $\zeta = 1.2357$ .

As  $\zeta$  is increased, the fidelity appears to approach a limit which is above  $1/2$ . Following Ref. 5, we conclude that the entanglement contained in the pair-coherent state is being utilized in the process. In addition, we argue that obtaining fidelity greater than  $1/2$  also rules out the existence of an extended hidden variable model suggested by Caves and Wódkiewicz.<sup>1</sup> This can be verified by calculating the teleportation fidelity using the “smeared-out” pair-coherent state  $W_{\text{kicked}}(\alpha, \beta) = Q(\alpha, \beta) \propto \exp(-|\alpha|^2 - |\beta|^2) |I_0(2\sqrt{\zeta}\alpha, \beta)|^2$ . This fidelity is also

included on Fig. 2, and shows that Alice and Bob obtain very low fidelities even for  $\zeta = 0$ , i.e. when they are sharing the vacuum. However, they have no other choice than applying the maximum “kicking” strength  $t = 1$ , since the case  $\zeta = 0$  is of zero measure, and for all other  $\zeta$  they must resort to the  $Q$  function of the original.<sup>20</sup>

## 5. Conclusions

In this paper we have proposed a modified setup of continuous variable teleportation which may allow for an experimental test of local realism in the teleportation process. Instead of the approach of Ref. 1, we replaced the quantum channel with a non-Gaussian, pair-coherent state. This permits us to use ordinary coherent states on the input and still rule out the possibility of a local hidden variable description. Teleporting coherent states has the additional advantage that the experimental determination of fidelity may be easier compared to some non-Gaussian input.

Our calculations yield a maximum fidelity  $\mathcal{F}_{\text{av}} = 0.75884$  for a particular pair-coherent state. This seems to be distant enough from  $1/2$ , therefore fidelities exceeding the classical limit could be observed experimentally also. We have also shown that this setup rules out the existence of an extended hidden-variable model<sup>1</sup> based on smearing out the participating quantum states.

## Acknowledgments

This work was partly supported by the National Research Fund of Hungary under contract Nos. T 068736 and T 049234. A. Gábris would also like to thank J. Janszky for useful discussions.

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