Acta Phys. Hung. B 23/1–2 (2005) 19–24

QUANTUM
ELECTRONICS

Controlled-NOT Gates for Four-Level Atoms in a Bimodal Cavity*

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Received 22 March 2005

Abstract. We present a construction of the full set of controlled-NOT gates for four-level atoms trapped in a bimodal cavity. The qubits are defined as the two ground states of every atom and the single photon subspace of the cavity Hilbert space. For the construction we employ the dispersive interaction of the ground states and the cavity, and also single-qubit operations. The possibility to implement the full set of controlled-NOT gates indicates that the system is suitable for universal Quantum Computation.

Keywords: quantum computing, cavity quantum electrodynamics, trapped atoms
PACS: 03.67.Lx, 42.50.Ct, 42.50.Pq, 42.50.Hz

* Based on the talk held at the *11th Central European Workshop on Quantum Optics (CEWQO11)*, 18–20 July 2004, Trieste, Italy.

1. Introduction

Interaction of trapped atoms or ions with a cavity is becoming subject of increasing interest in connection with quantum computing. Due to recent advances in experimental techniques of trapping and cavity fabrication [1–5] trapping of atoms inside cavities might soon become a reality. More-or-less common of quantum computing proposals regarding such systems is that the operation of multi-qubit gates rely on light-matter interaction rather than the interaction between the trapped particles [6–12]. The use of interaction between charged ions for quantum computation is well established, however, scalability appears to require significant modification of the original trapping apparatus [4]. Interactions between neutral atoms, on the

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other hand, are very weak and therefore require much more time for the complete operation than we can hope to preserve quantum coherence for.

In this paper we employ the dispersive interaction between four-level atoms and a bimodal cavity field, using the results of [13] to give a mathematical description of the complete system. To define the qubits, we use an invariant subspace of the effective Hamiltonian that consists of the ground states of the atoms and the single photon excitation states of the cavity. The structure of the resulting Hamiltonian is similar to that of describing the interaction of nuclear spins in a molecule which is employed in NMR quantum computing [14–16]. We develop quantum gates along the same lines as it is generally followed in NMR, and show the universality of the system by constructing controlled-NOT gates between every pairs of qubits.

2. Theoretical Model

We consider a system of N identical four-level atoms localized inside a bimodal cavity. The level scheme of the jth atom and its relation to the cavity excitations is depicted on Fig. 1. The cavity modes are characterized by their polarization and frequency. The frequencies of the $\sigma^{\pm}$ circular polarization is denoted by ω. The detunings of the cavity modes from the atomic transitions is $\Delta = \omega_0 - \omega$. The complete atoms–cavity Hamiltonian in the rotating wave approximation reads:

$$
\begin{aligned}
H = {} & \hbar\frac{\omega_0}{2}\sum_{j=1}^{N}\left(|e_-\rangle\langle e_-|_j - |g_+\rangle\langle g_+|_j\right) + \hbar\frac{\omega_0}{2}\sum_{j=1}^{N}\left(|e_+\rangle\langle e_+|_j - |g_-\rangle\langle g_-|_j\right) \\
& + \hbar\left(\omega^+ a_+^\dagger a_+ + \omega^- a_-^\dagger a_-\right) \\
& + \hbar\left(a_-\sum_{j=1}^{N} g_-|e_-\rangle\langle g_+|_j + \text{h.c.}\right) + \hbar\left(a_+\sum_{j=1}^{N} g_+|e_+\rangle\langle g_-|_j + \text{h.c.}\right) \qquad (1)
\end{aligned}
$$

In the dispersive limit ($\Delta \gg |g_\pm|$) an effective Hamiltonian can be derived using the fact that (1) can be regarded as a sum of two Hamiltonians each describing the interaction of a two level system with a single cavity mode. In the interaction picture this Hamiltonian depends only on the detuning (Δ), and possesses two invariant subspaces in the atomic part of the Hilbert space, one containing only excited and

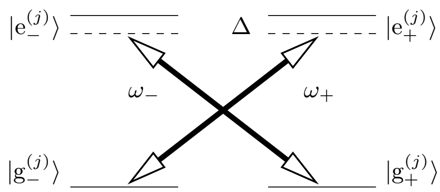

Fig. 1. Energy levels and cavity excitations

the other only ground states [13]. Therefore if the initial states are superpositions of $|g_\pm^{(j)}\rangle$ ground states, the final state must be a similar superposition. If we denote the dipole coupling constants $g_\pm = g$ (g real), in the interaction picture the effective Hamiltonian acting on the ground states can be written:

$$H_{\text{eff}} = \frac{\hbar}{2\Delta} N_z S_z, \tag{2}$$

with

$$S_z = \sum_{j=1}^{N} g^2 S_z^{(j)}, \tag{3}$$

$S_z^{(j)} = |g_+^{(j)}\rangle\langle g_+^{(j)}| - |g_-^{(j)}\rangle\langle g_-^{(j)}|$ and $N_z = a_+^\dagger a_+ - a_-^\dagger a_-$.

Further, we may restrict ourselves to the case when the cavity contains exactly one photon, i.e. the cavity state can be written as a superposition of $|10\rangle$ and $|01\rangle$, where the first number corresponds to a Fock state of the σ^+ and the second to the σ^- polarization mode. In other words, we use the polarization states of a photon to represent a quantum bit, a concept that has been used widely in the field, most remarkably for the experimental realization of quantum teleportation [17].

We now introduce new notations for the states of the system, namely

$$\begin{aligned} |0\rangle_0 &:= |10\rangle\,, \\ |1\rangle_0 &:= |01\rangle \end{aligned} \tag{4}$$

for the cavity, what is known as dual-rail representation, and

$$\begin{aligned} |0\rangle_j &:= |g_+^{(j)}\rangle\,, \\ |1\rangle_j &:= |g_-^{(j)}\rangle \end{aligned} \tag{5}$$

for the atomic states ($j = 1, \ldots, N$). Using these notations the effective Hamiltonian can be written as

$$H_{\text{eff}} = \frac{\hbar}{2\Delta} \sum_{j=1}^{N} g_j^2 \sigma_z^{(0)} \sigma_z^{(j)}, \tag{6}$$

where $\sigma_z^{(\alpha)} = |0\rangle\langle 0|_\alpha - |1\rangle\langle 1|_\alpha$ is a Pauli-z matrix for every $\alpha = 0, \ldots, N$.

In the remaining, Latin indices (e.g. i, j) shall always run from 1 to N, and Greek indices (e.g. α, β) from 0 to N, unless otherwise indicated.

3. Implementing Controlled-NOT Gates

In this section we shall show that the interaction described by (6) may be used to generate the universal CNOT gate. The required single-qubit operations could be implemented using external controls. For example, on the atoms by resonant Raman transitions, and on the cavity state by a technique similar to Refs. [18, 19].

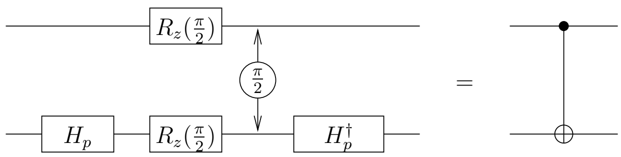

Fig. 2. Construction of CNOT gate using a J-coupling interaction

First, we recall that (6) is a special case of the NMR Hamiltonian [14–16] with $J_{ij} = J$ if $i = 0$ and $j > 0$, otherwise $J_{ij} = 0$. Since $[\sigma_z^{(\alpha)}, \sigma_z^{(\beta)}] = 0$ all terms in (6) commute, therefore the time evolution operator $U(t) = \exp\left[-(i/\hbar)H_{\text{eff}}t\right]$ may be written as a product of

$$U_{0k}^J(t) = \exp\left(-iJ\sigma_z^{(0)}\sigma_z^{(k)}t/2\right) \tag{7}$$

operators ($J = g^2/\Delta$). We shall refer to this operator as J-coupling term, borrowed from the NMR terminology. We write the time evolution prescribed by the total Hamiltonian as

$$U(t) = \exp\left[-(i/\hbar)H_{\text{eff}}t\right] = \prod_{j=0}^{N} U_{0k}^J. \tag{8}$$

Based on a technique used in NMR, we can cancel any of the J-coupling terms in the time evolution operator (8). This involves casting single-qubit rotations on some qubits in between time evolution governed by H_{eff}. The following identity lies in the heart of the technique:

$$U_\beta^x(\pi)U_{\alpha\beta}^J(t)U_\beta^x(\pi)U_{\alpha\beta}^J(t) = \mathbb{1}, \tag{9}$$

where $U_\alpha^x(\varphi)$ denotes the required rotation. We take this this rotation to be about the x axis: $U_\alpha^x(\varphi) = \exp\left(-i\sigma_x^{(\alpha)}\varphi/2\right)$, where $\sigma_x^{(\alpha)} = |0\rangle\langle 1|_\alpha + |1\rangle\langle 0|_\alpha$, the Pauli-$x$ operator.

Application of similar sequence of π rotations to the kth qubit will cancel the J-coupling term connecting the cavity and the kth atomic qubit:

$$\begin{aligned} U_k^x(\pi/2)U(t/2)U_k^x(\pi/2)U(t/2) &= U_k^x(\pi/2)U_{0k}^J(t)U_k^x(\pi/2)U_{0k}^J(t)\prod_{j\neq k} U_{0j}^J(t) \\ &= \prod_{j\neq k} U_{0j}^J(t), \end{aligned} \tag{10}$$

since $\left[U_k^x, U_{0,j}^J\right] = 0$ for each $j \neq k$. It is straight-forward to show that successive or simultaneous applications of similar sequence of π rotations on other atomic qubits can be used to cancel any number of terms in (6). In particular, $N-1$ terms may be cancelled leaving only one term, therefore H_{eff} can be used to realize time evolution described by a single U_{0k}^J. This operator is equivalent [20] to the controlled-NOT gate when $J_k t = \pi/2$, and can be used directly to generate it as it is depicted on Fig. 2.

The technique depicted on Fig. 2 utilizes two classes of single-qubit operators: rotations about the x and the z axes. Since the J-coupling interaction is symmetric with respect to the two qubits, the role of qubits in the resulting CNOT gate is determined by the single-qubit gates which precede and follow $U^J_{0k}(J_k^{-1}\pi/2)$. In particular, preceding the $U^J_{0k}(J_k^{-1}\pi/2)$ time evolution an $R_z(\pi/2)$ rotation has to be applied to the control qubit, and the sequence of H_p, and $R_z(\pi/2)$ to the target qubit, where H_p is a pseudo-Hadamard transformation given by

$$H_p = R_z(-\pi)R_x(\pi/2)R_z(\pi) = R_y(\pi/2). \tag{11}$$

After the J-coupling we apply the inverse of the H_p operation to the target qubit only. Note, however, that the R_z operations commute with U^J_{0k}.

We can use this technique to implement CNOT gates directly between the cavity qubit and any atomic qubit. In two steps, however, they can be used to effect CNOT gates between any atomic qubits also. One first constructs a SWAP gate from three CNOT gates using the well-known identity, then uses the SWAP gates to switch an atomic qubit onto the cavity and back, as depicted on Fig. 3 using the standard quantum circuit notations.

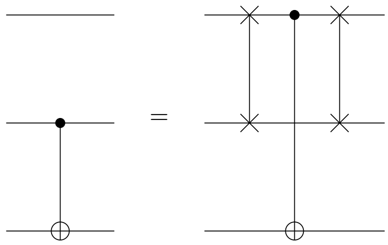

Fig. 3. Construction of a CNOT gate between two atomic qubits, using two SWAP gates

By noting that $R_x(-\pi/2) = R_x(\pi/2)^3$, we conclude that it is possible to exactly implement all CNOT gates in this system provided that we can perform on every qubit at least two single-qubit operations, $R_x(\pi/2)$ and $R_z(\pi/2)$.

4. Conclusions

In this paper we have demonstrated the possibility of generating the complete set of CNOT gates in an N qubit system, using a dispersive atom–cavity interaction. We encode our qubit in the ground state of four-level atoms, and into a single-photon excitation subspace of a bimodal cavity. The resulting interaction Hamiltonian is similar to the one valid for NMR systems, however, considerably simpler even for large number of atoms. Due to this simple scaling property of the interaction Hamiltonian, given that the requirements on trapping and individual addressing are met, the system could be scaled up to large qubit numbers.

Acknowledgments

This work was partly supported by the National Research Fund of Hungary under contract Nos. T 034484 and T 043079; the Marie Curie Programme of the European Commission; and by the Hungarian Ministry of Education under contract No. CZ-5/03.

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